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A first exploration of the split-interval coloring polytope

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Abstract

Given a graph $G = (V, E)$, a set of consecutive colors, and a demand vector $d \in \mathbb{Z}_+^{|V|}$, the interval coloring problem asks for an assignment of d_i consecutive colors to each vertex $i \in V$, in such a way that no two adjacent vertices are assigned the same color. Inspired by a situation arising in the allocation of ships to berths, in this work we propose to consider the *split-interval coloring problem*, which asks to assign at most two disjoint color intervals to each vertex in such a way that each vertex $i \in V$ receives a total of d_i colors. We explore a natural integer programming formulation for this NP-hard problem and its associated polytope. We state some relations to the interval coloring polytope, including lemmas allowing to translate valid inequalities between these two polytopes. We also present several valid inequalities and study conditions ensuring that these inequalities induce facets of the associated polytope.

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Keywords: interval coloring; berth allocation problem; polyhedral combinatorics

1. Introduction

Given a graph $G = (V, E)$, a *demand vector* $d \in \mathbb{Z}_+^{|V|}$, and a set $C = \{1, \ldots, c\}$ of consecutive so-called *colors*, the *interval coloring* problem (IC) consists in assigning a bounded integer interval $I_i \subseteq C$ of length $w(I_i) = d_i$ to each vertex $i \in V$, in such a way that $I_i \cap I_j = \emptyset$ whenever $ij \in E$ [6]. Interval coloring generalizes the classical vertex coloring problem and has been proposed as a theoretical model for several resource allocation problems, including bandwidth allocation [3, 15, 7, 10, 8] and berth allocation [1, 2, 12], among others.

In this work, we are interested in extensions of the classical interval coloring problem inspired by berth allocation scenarios, in which vessels have to be assigned time intervals to perform loading or unloading operations. A particular characteristic of real-world berth allocation is the possibility of splitting the service of a vessel into multiple, non-contiguous time intervals. In addition, certain pairs of vessels cannot be served simultaneously, even if they are assigned to different berths, due to shared resources such as private tugboats, cranes, access channels, or shared personnel. These shared resources introduce *incompatibility constraints* between specific pairs of vessels, making it necessary to ensure that their assigned intervals do not overlap in time. This leads to a generalized interval coloring

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problem in which each vertex (vessel) may be assigned multiple time intervals, subject to pairwise incompatibility restrictions.

While the possibility of splitting a vessel's service into multiple intervals offers additional flexibility, in practice such preemptions are costly and operationally challenging. Interrupting a vessel's service typically requires the use of tugboats, coordination with pilots, and rescheduling of quay cranes and personnel. As a result, most port operations tolerate at most one interruption per vessel, allowing a maximum of two non-contiguous service intervals. Splitting the service into more than two intervals would add significant complexity and operational burden, which in most cases would not be justified by the potential gains.For this reason, in this work we focus on the case in which each vessel may be assigned at most two intervals, thus balancing between modeling flexibility and operational realism. The graph modeling this problem represents each vessel with a distinct vertex connecting two vertices with an edge whenever the corresponding vessels cannot be served at the same time—either because they require the same berth or share critical resources such as tugboats, cranes, or access channels. The set of available colors corresponds to a discrete time horizon, where each color represents a unit of time during which service can be scheduled.

These observations lead us to state a generalization of the classical interval coloring problem, in which each vertex can be assigned more than one interval. As a starting point, we propose to explore what we suggest to call the *split-interval coloring* problem (SIC), in which each vertex can be assigned up to two intervals, in such a way that the sum of the lengths of the intervals assigned to the vertex i is d_i, for $i \in V$. Any pair of intervals assigned to adjacent vertices cannot overlap.

Definition 1 (split-interval coloring problem). *Given a graph $G = (V, E)$, a* demand vector $d \in \mathbb{Z}_+^{|V|}$*, and a set $C = \{1, \ldots, c\}$ of consecutive* colors*, determine whether each vertex $i \in V$ can be assigned two non-overlapping bounded integer intervals intervals $I_i, I'_i \subseteq C$ in such a way that $w(I_i) + w(I'_i) = d_i$ and any two intervals assigned to adjacent vertices do not overlap.*

We can represent a split-interval coloring by resorting to a graph G' obtained from G by duplicating all the vertices from G, replicating the original edges and adding an edge between each vertex and its duplicate. Formally, the graph $G' = (V', E')$ has vertex set $V' := V \cup \{i' : i \in V\}$, where we assume that $i' \neq j$ for every $i, j \in V$, and edge set

$$\begin{aligned} E' &= \{ij : ij \in E\} \cup \{i'j : ij \in E\} \\ &\cup \{ij' : ij \in E\} \cup \{i'j' : ij \in E\} \cup \{ii' : i \in V\}. \end{aligned}$$

For $i \in V$, we refer to $i' \in V' \setminus V$ as the *twin vertex* of i. A split-interval coloring of G corresponds to an interval coloring of G' such that the lengths of the intervals assigned to the vertices i and i' sum exactly d_i, for every $i \in V$.

The split-interval coloring problem is NP-complete since it contains, e.g., the classical vertex coloring problem on graphs as a restricted problem (namely, when $d_i = 1$ for every $i \in V$, there exists a split-interval coloring of G if and only if $\chi(G) \leq c$). This motivates the application of integer programming techniques to this problem, and such a task is started in the remainder of this work.

Generalizations of the interval coloring problem allowing each vertex to be assigned multiple intervals have already been considered in the literature (see, e.g., [5, 14, 13]). Such generalizations correspond to tasks that may be split across time or machines, with the additional consideration that preemptions are costly or constrained. In this work we explore these issues in the particular setting described in this section, and with a special interest in the polyhedral structure of an associated integer programming formulation.

2. The split-interval coloring polytope

We can provide an integer programming formulation for SIC as follows. For each vertex $i \in V'$, we introduce the integer variables l_i and r_i in such a way that $[l_i, r_i)$ represents the interval assigned to the vertex i. This amounts to defining four integer variables for each vertex $i \in V$, namely the integer variables l_i, r_i, $l_{i'}$, and $r_{i'}$, in such a way that $[l_i, r_i)$ and $[l_{i'}, r_{i'})$ represent the two intervals assigned to the vertex i (we denote by $[a, b) := \{a, a+1, \ldots, b-1\}$ the interval of integer values spanning from a to $b-1$, for any $a, b \in \mathbb{Z}$, $a \leq b$, and this interval is empty if $a = b$). We represent the assignment of only one interval to the vertex $i \in V$ (which has, therefore, length d_i) by setting the other interval to have null length. For $ij \in E'$, $i < j$, we introduce the binary variable x_{ij} representing whether $r_i \leq l_j$ or not,

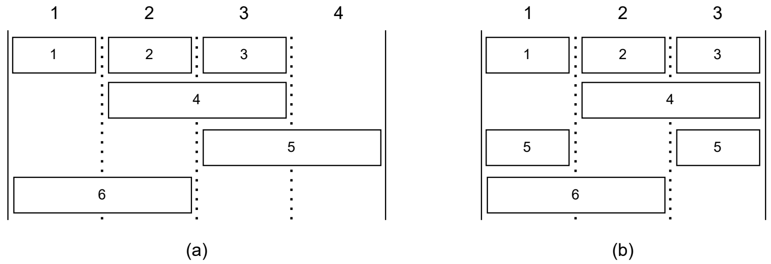

Fig. 1. Let $G = (\{1, \ldots, 6\}, \{12, 23, 13, 14, 25, 36\})$ with $d_i = 1$ for $i = 1, 2, 3$ and $d_i = 2$ for $i = 4, 5, 6$. Subfigure (a) shows an optimal solution in $P_{IC}(G, d, c)$ with four colors (no solution with three colors exists, since the vertex in the clique $\{1, 2, 3\}$ receiving the color 2 forces its neighbor outside the clique to use the color 4), whereas Subfigure (b) shows a solution in $P_{SIC}(G, d, c)$ with three colors for the same graph. Each column represents a color, and the horizontal boxes represent the color intervals assigned to the vertices.

and the binary variable x_{ji} representing whether $r_j \leq l_i$ or not. In other words, $x_{ij} = 1$ if the interval assigned to i is located before the interval assigned to j. These variables cover all the edges in G', so we have the binary variables x_{ij}, x_{ji}, $x_{i'j}$, $x_{ji'}$, $x_{ij'}$, $x_{j'i}$, $x_{i'j'}$, and $x_{j'i'}$ for each edge $ij \in E$, and the two binary variables $x_{ii'}$ and $x_{i'i}$ for each vertex $i \in V$.

In this setting, an assignment of values to these variables represents a split-interval coloring if the following constraints are satisfied.

$$(r_i - l_i) + (r_{i'} - l_{i'}) = d_i \qquad \forall i \in V \tag{1}$$
$$r_i \leq l_j + c\,(1 - x_{ij}) \qquad \forall ij \in E', i < j \tag{2}$$
$$r_j \leq l_i + c\,(1 - x_{ji}) \qquad \forall ij \in E', i < j \tag{3}$$
$$x_{ij} + x_{ji} = 1 \qquad \forall ij \in E', i < j \tag{4}$$
$$0 \;\leq\; l_i \leq r_i \;\leq\; c \qquad \forall i \in V' \tag{5}$$
$$l_i, r_i \in \mathbb{Z}_+ \qquad \forall i \in V' \tag{6}$$
$$x_{ij}, x_{ji} \in \{0, 1\} \qquad \forall ij \in E', i < j \tag{7}$$

Constraints (1) express that the total length of the intervals assigned to each vertex $i \in V$ from G must obey the demand d_i. Together with constraints (5), this allows any interval to have null length and, in this case, we consider that a single interval is assigned to the vertex i. Constraints (2) state that if $x_{ij} = 1$ then the interval assigned to i must be located before the interval assigned to j, for $ij \in E'$, $i < j$, and constraints (3) handle the symmetrical situation. Together with constraints (4), these constraints forbid overlapping between intervals associated to adjacent vertices in G'. Finally, constraints (5)-(7) impose the domains for the variables.

Definition 2. *We define $P_{SIC}(G, d, c)$ to be the convex hull of the points $(l, r, x) \in \mathbb{R}^{2|V'|+2|E'|}$ satisfying constraints (1)-(7).*

We denote by $\chi_{IC}(G, d)$ the minimum number of colors c such that G admits an interval coloring with c colors. Similarly, we call $\chi_{SIC}(G, d)$ to the minimum number of colors c such that G admits a split-interval coloring with c colors (i.e., such that $P_{SIC}(G, d, c)$ is nonempty). We clearly have $\chi_{SIC}(G, d) \leq \chi_{IC}(G, d)$, since any interval coloring is also a split interval coloring. Furthermore, we may have $\chi_{SIC}(G, d) < \chi_{IC}(G, d)$, as Figure 1 shows. The parameter $\chi_{SIC}(G, d)$ can be calculated by slightly modifying the formulation (1)-(7), namely by turning c into a decision variable and adding the minimization of the latter as the objective function. The nonlinearities introduced by this procedure can be avoided by replacing c in constraints (2)-(3) with an upper bound M for $\chi_{SIC}(G, d)$ (as, e.g., $M = \sum_{i \in V} d_i$, which gives a trivial upper bound for $\chi_{SIC}(G, d)$).

Proposition 1. *If $c > \chi_{SIC}(G, d)$, then* $\dim(P_{SIC}(G, d, c)) = 3|V| + |E'|$.

The particular symmetry of the set of colors allows us to state the following lemma, which is shared with the classical interval coloring polytope (see e.g., [7]). We denote by **1** the all-ones vector of appropriate dimension.

Lemma 1 (Symmetry Lemma). *Let $\alpha^\top l + \beta^\top r + \gamma^\top x \leq \delta$ be a valid (resp. facet-inducing) inequality for $P_{SIC}(G,d,c)$. Then, $\beta^\top(c\mathbf{1} - l) + \alpha^\top(c\mathbf{1} - r) + \gamma^\top(\mathbf{1} - x) \leq \delta$ is a valid (resp. facet-inducing) inequality for $P_{SIC}(G,d,c)$.*

The polytope $P_{SIC}(G,d,c)$ admits further symmetries since any assignment of two intervals to a vertex can be represented in two ways. We can avoid this symmetry by adding either the symmetry-breaking constraint $r_i \leq l_{i'}$ or the symmetry-breaking constraint $r_i - l_i \geq r_{i'} - l_{i'}$ for every $i \in V$. However, this addition may complicate the structure of the polytope, and in this initial polyhedral study we keep the original formulation (1)-(7) giving rise to the polytope $P_{SIC}(G,d,c)$.

We denote by $P_{IC}(G,d,c)$ the interval coloring polytope over the l- and the x-variables, i.e., the convex hull of all points $(l,x) \in \mathbb{Z}^{|V|} \times \{0,1\}^{2|E|}$ representing interval colorings of G, where the l- and the x-variables have the same interpretation as in the formulation (1)-(7). When referring to this polytope, we assume $r_i = l_i + d_i$ as a notational convenience, for $i \in V$. The polytope $P_{IC}(G,d,c)$ has been extensively studied in previous works [7, 10, 8, 9].

In the remainder of this section, we propose tools that transform valid inequalities for $P_{IC}(G,d,c)$ into valid inequalities for $P_{SIC}(G,d,c)$. As an example, for a given edge $ij \in E$ consider the inequality

$$d_i x_{ij} \leq l_j, \tag{8}$$

which is valid for $P_{IC}(G,d,c)$. Clearly, inequality (8) is not valid for $P_{SIC}(G,d,c)$, as the vertex i could receive strictly less than d_i colors (the remainder of its demand being assigned to its twin i'). However, we can replace d_i with the actual number of colors assigned to vertex i, which is exactly $(r_i - l_i)$, thus obtaining the nonlinear inequality

$$(r_i - l_i)x_{ij} \leq l_j, \tag{9}$$

which is valid for $P_{SIC}(G,d,c)$ (we shall deal later with the fact that the obtained inequality is nonlinear). The "replacement" of d_i by $(r_i - l_i)$ in the syntactic expression of the inequality (8) is an informal way of explaining the proposed procedure. In order to formalize this procedure, we introduce the following definitions.

Definition 3. *Fix a graph $G = (V,E)$ and a number c of colors.*

(a) Consider an arbitrary function $f : \mathbb{R}^{|V|} \times \mathbb{R}^{2|E|} \times \mathbb{R}^{|V|} \to \mathbb{R}$. We say that f is a valid function for P_{IC}, G, and c *if, for every $d \in \mathbb{Z}_+^{|V|}$, we have that $f(l,x,d) \leq 0$ holds for every point $(l,x) \in P_{IC}(G,d,c)$.*

(b) Consider an arbitrary function $f : \mathbb{R}^{|V'|} \times \mathbb{R}^{|V'|} \times \mathbb{R}^{2|E'|} \times \mathbb{R}^{|V|} \to \mathbb{R}$. We say that f is a valid function for P_{SIC}, G, and c *if, for every $d \in \mathbb{Z}_+^{|V|}$, we have that $f(l,r,x,d) \leq 0$ holds for every point $(l,r,x) \in P_{SIC}(G,d,c)$.*

For example, the inequality (8) corresponds to the valid function $f(l,x,d) = d_i x_{ij} - l_j$ for P_{IC}, G, and c, whereas the inequality (9) corresponds to the valid function $f(l,r,x,d) = (r_i - l_i)x_{ij} - l_j$ for P_{SIC}, G, and c. Throughout this section we will call $f(l,x,d) \leq 0$ a *valid inequality* if f is a valid function for P_{IC}, G, and c, and we make an analogous definition for P_{SIC}. It is important to note that the demand vector d is also a parameter of the function f, and the definition of "valid function" needs the inequality to be satisfied by every point in the polytope associated to every demand vector.

We now state one of the main lemmas in this section, which explores the replacement of all appearances of d_i by $(r_i - l_i)$ in a valid function, for every $i \in V$. If $l \in \mathbb{R}^{|V'|}$, we define $p(l)$ to be the projection of l onto the variables $\{l_i\}_{i\in V}$, namely the variables associated with the original vertices (not the twins). If $x \in \mathbb{R}^{2|E'|}$, we define $p(x)$ to be the projection of x onto the variables $\{x_{ij}, x_{ji}\}_{ij\in E, i<j}$, namely the variables associated with the original edges (i.e., not involving any twin vertex).

Lemma 2 (Conversion Lemma). *Fix a graph $G = (V,E)$ and a number c of colors.*

(a) If $f : \mathbb{R}^{|V|} \times \mathbb{R}^{2|E|} \times \mathbb{R}^{|V|} \to \mathbb{R}$ is a valid function for P_{IC}, G, and c, then $f'(l, r, x, d) := f(p(l), p(x), p(r) - p(l))$ is a valid function for P_{SIC}, G, and c.

(b) Fix $d \in \mathbb{Z}_+^{|V|}$. If (i) $c > \chi_{SIC}(G, d)$, (ii) $f : \mathbb{R}^{|V|} \times \mathbb{R}^{2|E|} \times \mathbb{R}^{|V|} \to \mathbb{R}$ is a valid function for P_{IC}, G, and c, (iii) there exist $|V| + |E|$ affinely independent points in $P_{IC}(G, d, c)$ satisfying $f(l, x, d) = 0$, (iv) there exists some vertex $k \in V$ such that no variable associated with k appears in f, and (v) for every $j \in V$ there exists $(l^j, r^j, x^j) \in P_{SIC}(G, d, c)$ satisfying $f'(l^j, r^j, x^j, d) = 0$ and such that $r_{j'} - l_{j'} > 0$, then there exist $3|V| + |E'|$ affinely independent points in $P_{SIC}(G, d, c)$ satisfying $f'(l, r, x, d) = 0$.

Part (a) of the Conversion Lemma follows from the fact that the intervals assigned to the original vertices (i.e., not the twins) provides an interval coloring of a graph with potentially different demands for the vertices. Part (b) is a quite technical assertion, which will be useful in Section 3 in order to explore facetness properties of valid inequalities. Its proof comes from manipulating a set of affinely independent points satisfying the inequality with equality. The valid inequalities generated by the Conversion Lemma are usually nonlinear, so we are interested in procedures to linearize them. The following lemma will be the main tool to accomplish this task, based on the classical McCormick envelopes for bilinear functions [11].

Lemma 3 (Flattening Lemma). *Let $i \in V'$ and $e \in E'$. If $f(l, r, x, d) + \alpha(r_i - l_i)x_e \le 0$ is a valid inequality for $P_{SIC}(G, d, c)$ and $\alpha > 0$, then the inequality $f(l, r, x, d) + \alpha(r_i - l_i) - \alpha d_i(1 - x_e) \le 0$ is also valid for $P_{SIC}(G, d, c)$. Furthermore, if the resulting inequality is linear and there exist $\dim(P_{SIC}(G, d, c))$ affinely independent points satisfying the original inequality with equality and also satisfying $x_e = 0 \Rightarrow r_i = l_i + d_i$, then the new inequality induces a facet of $P_{SIC}(G, d, c)$.*

Corollary 1. *If $f(l, r, x, d) + \alpha(r_i - l_i)x_{ij} + \alpha(r_{i'} - l_{i'})x_{i'j} \le 0$ is a valid inequality for $P_{SIC}(G, d, c)$ and $\alpha > 0$, then $f(l, r, x, d) + \alpha d_i(x_{ij} + x_{i'j} - 1) \le 0$ is a valid inequality for $P_{SIC}(G, d, c)$.*

The following lemma enables a similar result when the nonlinearity is given by the subtraction of two x-variables, a situation that is usual in this context. Their proofs are also based on the classical McCormick envelopes for bilinear functions.

Lemma 4 (Reverse Flattening Lemma). *If $f(l, r, x, d) + \alpha(r_i - l_i)x_e \le 0$ is a valid inequality for $P_{SIC}(G, d, c)$ and $\alpha < 0$, then $f(l, r, x, d) + \alpha d_i x_e \le 0$ is a valid inequality for $P_{SIC}(G, d, c)$.*

Corollary 2 (Double Flattening Lemma). *If $f(l, r, x, d) + \alpha(r_i - l_i)(x_e - x_{e'}) \le 0$ is a valid inequality for $P_{SIC}(G, d, c)$ and $\alpha > 0$, then $f(l, r, x, d) + \alpha(r_i - l_i) - \alpha d_i(1 - x_e) - \alpha d_i x_{e'} \le 0$ is a valid inequality for $P_{SIC}(G, d, c)$.*

Finally, the fact that each vertex and its twin have symmetrical roles in the formulation implies the following result.

Lemma 5 (Replacement Lemma). *If $\alpha^\intercal l + \beta^\intercal r + \gamma^\intercal x \le \delta$ is a valid inequality for $P_{SIC}(G, d, c)$ and $i \in V$, then the inequality obtained by replacing i by i' and vice-versa is also a valid inequality for $P_{SIC}(G, d, c)$. Furthermore, if $\alpha^\intercal l + \beta^\intercal r + \gamma^\intercal x \le \delta$ induces a facet of $P_{SIC}(G, d, c)$ then the resulting inequality also induces a facet of $P_{SIC}(G, d, c)$.*

3. Valid inequalities

In this section we introduce several families of valid inequalities and we identify conditions ensuring that these inequalities induce facets of $P_{SIC}(G, d, c)$, based on the results presented in the previous section. For $i \in V$, we define $N_G(i) := \{j \in V : ij \in E\}$ to be the set of neighbors in G of the vertex i, and we define $N_{G'}(i)$ similarly for $i \in V'$ and the graph G'. For $K \subseteq V'$, we define $d(K) := \sum_{k \in K} d_k$. We say that a set of vertices is a clique if the vertices in the set are pairwise adjacent.

Theorem 1. *If $i \in V'$ and $K \subseteq N_{G'}(i)$ is a clique in G', then the inequality*

$$l_i \ge \sum_{k \in K} (r_k - l_k) - d_k x_{ik} \tag{10}$$

is valid for $P_{SIC}(G, d, c)$*. If, furthermore, K does not contain a vertex and its twin and* $c > \chi_{IC}(G, d) + \max_{k \in K} d_k + \max_{j \in V' \setminus K} d_j$*, then this inequality induces a facet of* $P_{SIC}(G, d, c)$*.*

When $K \cup \{i\} \subseteq V$, inequality (10) comes from the application of the Conversion Lemma, the Flattening Lemma, and the Replacement Lemma to the so-called *clique inequalities* for the interval coloring polytope [10]. It is interesting to note that no additional hypothesis on the graph G are needed in order to (10) to define a facet, as opposed to a similar situation in the interval coloring polytope. This is due to the fact that the splitting of the demands allows a greater flexibility in constructing solutions in the face defined by the inequality, thus giving rise to a stronger inequality. Besides using the chain of auxiliary lemmas mentioned before, the validity for (10) can be proved from first principles by a direct argument, whereas facetness follows from the enumeration of a suitable number of affinely independent points in the face defined by the inequality.

Theorem 2. *If* $ij \in E'$*,* $j \neq i'$*, and* $K \subset N_{G'}(i) \cap N_{G'}(j)$ *is a clique in* G'*, then the inequality*

$$l_j - r_i \geq \sum_{k \in K} (r_k - l_k) - d_k(2 - x_{ik} - x_{kj}) - (c - d(K))(1 - x_{ij}) \tag{11}$$

is valid for $P_{SIC}(G, d, c)$*. If, furthermore, K does not contain a vertex and its twin and* $c > \chi_{IC}(G, d) + \max_{k \in K} d_k + \max_{j \in V' \setminus K} d_j$*, then this inequality induces a facet of* $P_{SIC}(G, d, c)$*.*

The inequality (11) is obtained by applying the Flattening Lemma and the Replacement Lemma to the valid inequality

$$l_j - r_i \geq \sum_{k \in K} (r_k - l_k)(x_{ik} - x_{jk}) - (c - d(K))(1 - x_{ij}), \tag{12}$$

which in turn comes from the application of the Conversion Lemma to the so-called *double clique inequalities* for the interval coloring polytope [10]. Again, both validity and facetness for (11) can be established from first principles by direct arguments.

Theorem 3. *Let* $ij \in E$ *and let* $K \subseteq N_G(i) \cap N_G(j)$ *be a clique in G. The inequality*

$$l_i + l_{i'} + l_j + l_{j'} \;\geq\; \min\{d_i, d_j\} + \sum_{k \in K} (r_k - l_k) - d_k x_{ik} \tag{13}$$

is valid for $P_{SIC}(G, d, c)$*. If, furthermore,* $d_i = d_j$ *and* $c > d_i + d_j + \chi_{SIC}(G \setminus \{i, j\}, d)$*, then (13) induces a facet of* $P_{SIC}(G, d, c)$*.*

The validity proof of Theorem 3 comes from the fact that the intervals associated to the vertices $\{i, i', j, j'\}$ cannot overlap (since $ij \in E$), hence at least one of them must start at $\min\{d_i, d_j\}$ or later. The facetness proof is obtained by constructing a suitable number of affinely independent points in the face defined by (13).

In the following result we seek valid inequalities capturing the divisible nature of the intervals in the split-interval coloring problem. To this end, we consider a structure in G forcing at least one interval to be split into two nonempty intervals.

Theorem 4. *Let* $ij \in E$*, and let* $K \subseteq N_G(j) \setminus \{i\}$ *be a clique in G such that* $d_i + d_j + d(K) > c$*. The inequality*

$$x_{ij} + x_{i'j} \;\leq\; 1 + \sum_{k \in K} (x_{kj} + x_{k'j}) \tag{14}$$

is valid for $P_{SIC}(G, d, c)$*. If, furthermore,* $c > \chi_{SIC}(G, d)$ *and* $c \geq \chi_{SIC}(G \setminus \{i, k\}, d) + \max\{d_i, d_k\}$ *for every* $k \in K$ *then (14) induces a facet of* $P_{SIC}(G, d, c)$*.*

The hypothesis $d_i + d_j + d(K) > c$ ensures that if $x_{ij} = x_{i'j} = 1$, then at least one interval associated with the vertices in K must be located before j (since otherwise the number of colors is not sufficient to accommodate all these intervals). Again, facetness is proved by constructing a suitable number of affinely independent points satisfying (14) with equality. The conditions ensuring that (14) induces a facet are quite tight, so we would not expect this inequality to induce a facet in general. However, this inequality could be useful in order to rule out infeasible solutions when there is a clique imposing conditions on the feasibility of the whole instance.

Theorem 5. *Let* $i, j, k, t \in V$ *be four distinct vertices such that* $\{ij, jk, kt\} \subseteq E$*. Then, the inequalities*

$$l_i \geq (r_j - l_j) - (d_j + d_k)(1 - x_{ji}) + (r_k - l_k) - d_k(1 - x_{kj}), \tag{15}$$

$$l_i \geq (r_j - l_j) - (d_j + d_k + d_t)(1 - x_{ji}) + (r_k - l_k) - (d_k + d_t)(1 - x_{kj}) + (r_t - l_t) - d_t(1 - x_{tk}) \tag{16}$$

are valid for $P_{SIC}(G, d, c)$*. Furthermore, if* $c > \chi_{IC}(G, d)$ *then both inequalities induce facets of* $P_{SIC}(G, d, c)$*.*

Theorem 5 arises from a partial application of the Conversion Lemma and the Flattening Lemma to the following *path inequalities* [4] for the interval coloring polytope:

$$l_i \geq (d_j + d_k)x_{ji} + d_k x_{kj} - d_k, \tag{17}$$

$$l_i \geq (d_j + d_k + d_t)x_{ji} + (d_k + d_t)x_{kj} + d_t x_{tk} - (d_k + 2d_t). \tag{18}$$

4. Concluding remarks

In this work we have started an initial polyhedral exploration of a natural formulation of the split interval coloring problem, an extension of the classical interval coloring problem motivated by observations coming from the berth allocation literature. There are many contact points between berth allocation problems and interval coloring, and we believe that it could be worthwhile to further explore these relations. This could result in practical improvements for berth allocation problems and, conversely, interesting theoretical questions for the interval coloring community.

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