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An initial polyhedral study of the DR-AOV formulation for the routing and spectrum allocation problem

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Abstract

The routing and spectrum allocation (RSA) problem is a critical challenge in optical networks, in which the objective is to assign a path and a set of contiguous frequency slots to each demand, meeting technical constraints given by the network infrastructure. As a key solution to managing large-scale data traffic in such networks, RSA has gained significant attention in the last years. One of the most effective integer programming formulations for RSA is the so-called DR-AOV model, and it is relevant to gain both theoretical and practical insights on this formulation. In this work, we tackle the first of these by starting a polyhedral study of the convex hull of the feasible solutions of the DR-AOV model. We identify general properties of this polytope, we establish relations to interval coloring polytopes, and we present several families of facet-inducing inequalities.

Keywords: routing and spectrum allocation; integer programming; facets

1. Introduction

The *routing and spectrum allocation* (RSA) problem is one of the most promising solutions in the operation of flex-grid optical networks which aim to efficiently handle high data traffic. This NP-hard problem involves determining optimal routes and frequency slot allocations for a set of data transmission demands within the network, subject to technical constraints [3, 14].

Formally, consider a directed graph $G = (V, E)$ representing an optical fiber network and a set of demands $D = \{1, \ldots, n\}$, where each demand $d \in D$ is defined by a source node $s(d) \in V$, a target node $t(d) \in V$ (with $s(d) \neq t(d)$) and a required volume $v(d) \in \mathbb{Z}_+$. In addition, the optical network has a fixed number $\bar{s} \in \mathbb{Z}_+$ of available frequency slots, so the set of available slots is $S := \{0, \ldots, \bar{s} - 1\}$. The volume of each demand specifies the number of frequency slots it requires. A *lightpath for* $d \in D$ is a path in G from $s(d)$ to $t(d)$ together with a sequence of $v(d)$ consecutive frequency slots in S. In this setting, RSA asks to find a lightpath for each demand in such a way that if any two lightpaths share an arc then their slot intervals are disjoint.

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RSA is strongly connected with *interval coloring* (also called *chromatic scheduling* in the literature). Given a graph and a weight associated with each vertex, the interval coloring problem asks for an assignment of a set of consecutive colors to each vertex, in such a way that adjacent vertices receive non-overlapping intervals [9]. RSA can be seen as an interval coloring problem over the graph whose edges are pairs of demands that have been assigned (edge–) intersecting paths.

While the polyhedral aspects of interval coloring have been studied in the literature [5, 11, 10, 12], the theoretical and polyhedral aspects of RSA have not been extensively studied. The complexity of this problem applied to particular graph families is still unknown, and there is not much in the literature on the facetness of valid inequalities for different formulations of the problem. However, over the past few years, several approaches to practically solve RSA have been proposed, with numerous formulations resulting in a range of solution strategies [7, 8, 13, 15, 6, 4]. In [2], several integer programming models for RSA were introduced and evaluated through computational experiments. Among the models presented therein, the so-called *DR-AOV formulation* attained the best overall performance, so in this work we focus our attention on the theoretical aspects of this formulation.

The DR-AOV formulation includes the following variables. For each demand $d \in D$ and each arc $e \in E$, the binary variable y_{de} indicates whether or not the demand d is routed through the arc e. For any two demands $d, d' \in D$, $d \neq d'$, the binary variable $x_{dd'}$ is equal to 1 if the demands d and d' share an arc and if the last slot assigned to d is smaller than the first slot assigned to d'. Furthermore, for each demand $d \in D$, the integer variable l_d represents the first slot of the spectrum interval assigned to d, hence this demand is assigned the slot interval $\{l_d, \ldots, l_d + v(d) - 1\}$. For $u \in V$, we define $\delta^-(u) := \{v_1v_2 \in E : v_2 = u\}$ to be the set of arcs entering u, and we define $\delta^+(u) := \{v_1v_2 \in E : v_1 = u\}$ to be the set of outgoing arcs from u. In this setting, the DR-AOV formulation is the following integer programming model.

$$\min \sum_{d\in D} \sum_{e\in E} y_{de} \tag{1}$$

$$\sum_{e\in\delta^-(u)} y_{de} - \sum_{e\in\delta^+(u)} y_{de} = \begin{cases} -1 & \text{if } u = s(d) \\ 1 & \text{if } u = t(d) \\ 0 & \text{otherwise} \end{cases} \qquad \forall u \in V, \forall d \in D, \tag{2}$$

$$x_{dd'} + x_{d'd} \geq y_{de} + y_{d'e} - 1 \qquad \forall d, d' \in D, d \neq d', \forall e \in E, \tag{3}$$

$$l_d + v(d) \leq l_{d'} + \bar{s}(1 - x_{dd'}) \qquad \forall d, d' \in D, d \neq d', \tag{4}$$

$$0 \leq l_d \leq \bar{s} - 1 \qquad \forall d \in D, \tag{5}$$

$$y_{de} \in \{0, 1\} \qquad \forall d \in D, \forall e \in E, \tag{6}$$

$$x_{dd'} \in \{0, 1\} \qquad \forall d, d' \in D, d \neq d'. \tag{7}$$

Among the several objective functions that have been proposed for RSA, we choose in (1) to minimize the sum of all path lengths. Constraints (2) ensure flow conservation, defining a path from the source node to the target node for each demand. Constraints (3) enforce that if the demands d and d' use the same arc on their paths then $x_{dd'} + x_{d'd} \geq 1$, thus implying that the associated slot intervals do not overlap, for $d, d' \in D$, $d \neq d'$. Constraints (4) specify that if $x_{dd'} = 1$ then the last slot assigned to d must be smaller than the first slot assigned to d', for $d, d' \in D$, $d \neq d'$. Finally, constraints (5) bound the volume of all demands, and constraints (6) and (7) ensure that both the x- and the y-variables are binary.

In this work, we present a polyhedral study of the convex hull of the set of feasible solutions of the DR-AOV model. Our main objective is to fill a gap in the existing literature of RSA, which has focused on algorithms for solving RSA in practice but has devoted a comparatively smaller effort to understanding the theoretical properties of this problem. By addressing these aspects, our aim is to contribute theoretical insights that may help to understand the polyhedral side of RSA, thus offering a foundation for further practical methods.

The remainder of this work is organized as follows. Section 2 presents the polytope associated with the formulation DR-AOV and explores general properties of this polytope. Section 3 presents families of facet-inducing inequalities, taken on the one hand from the formulation and on the other hand from the relationship between RSA and interval coloring. Finally, Section 4 presents conclusions and lines for future research.

2. The DR-AOV polytope

In this section, we formally define the polyhedron associated with the set of feasible solutions of the model under study. After the definition, we proceed to present some combinatorial characteristics and properties which will be used to calculate the dimension of the polyhedron.

Definition 1. *We define the polyhedron $P(G, D, \bar{s})$ to be the convex hull of all solutions $(y, l, x) \in \mathbb{Z}^{|D||E|} \times \mathbb{Z}^{|D|} \times \mathbb{Z}^{|D|^2-|D|}$ satisfying constraints (2)-(7).*

The polytope $P(G, D, \bar{s})$ has a special symmetry property. For each solution $z = (y, l, x) \in P(G, D, \bar{s})$, there exists a symmetrical solution $\bar{z} = (y, \bar{l}, \bar{x})$ regarding the spectrum allocation, where $\bar{l}_d := \bar{s} - v(d) - l_d$ for every $d \in D$, and $\bar{x}_{dd'} := x_{d'd}$ for every $d, d' \in D$, $d \neq d'$. Such a symmetric solution corresponds to mirroring the slot assignment with respect to $(\bar{s} - 1)/2$. This symmetry in the solutions that define $P(G, D, \bar{s})$ allows us to obtain the following result. We define $\mathbf{v} \in \mathbb{Z}_+^{|D|}$ to be the vector whose d-th coordinate is $v(d)$, for $d \in D$. Also, define $\mathbf{1} = (1, \ldots, 1) \in \mathbb{Z}^{|D|}$ to be the all-ones vector with $|D|$ entries.

Theorem 1. *If $\alpha^\mathsf{T} y + \beta^\mathsf{T} l + \gamma^\mathsf{T} x \leq \pi_0$ is a valid (resp. facet-inducing) inequality for $P(G, D, \bar{s})$, then $\alpha^\mathsf{T} y - \beta^\mathsf{T} l + \hat{\gamma}^\mathsf{T} x \leq \pi_0 - \beta^\mathsf{T}(\bar{s}\mathbf{1} - \mathbf{v})$ is a valid (resp. facet-inducing) inequality for $P(G, D, \bar{s})$, where $\hat{\gamma}_{dd'} := \gamma_{d'd}$ for every $d, d' \in D$, $d \neq d'$.*

We now explore the polyhedral structure of $P(G, D, \bar{s})$. To this end, we introduce some definitions that will be useful throughout this section.

Definition 2. *We define the set of* edge-sharing demands *$L(G, D, \bar{s}) \subseteq D \times D$ associated with G, D, and $\bar{s}$ to be the set of pairs of demands $(d, d') \in D \times D$, $d < d'$, such that in every feasible solution of $P(G, D, \bar{s})$, the paths associated to d and d' share at least one arc.*

The set $L(G, D, \bar{s})$ can be seen as a set of *mandatory conflicting demands*, i.e., pairs of demands to which disjoint sets of slots must always be assigned, since there is no way to avoid an intersection between their associated paths. Therefore, if $(d, d') \in L(G, D, \bar{s})$ then the sequence of slots assigned to d must always be non-overlapping with the sequence of slots assigned to d', thus implying $x_{dd'} + x_{d'd} = 1$. This information is important for our purposes since the condition $x_{dd'} + x_{d'd} = 1$ provides an equation satisfied by all points in $P(G, D, \bar{s})$, thus bounding the dimension of this polytope. When the context is clear, we shall refer to the set $L(G, D, \bar{s})$ as L in order to simplify the notation.

Figure 1 presents an example of the construction of the set $L(G, D, \bar{s})$. Let G be the depicted graph, $D = \{d_1 = (s_1, t_1, 4), d_2 = (s_2, t_2, 2), d_3 = (s_3, t_3, 2)\}$, and $\bar{s} = 4$. In this case, demand d_1 will be assigned either the arc e_1 or e_2, and since there are not enough slots, d_2 and d_3 will not be able to use the arc assigned to d_1. Thus, regardless of the arc assigned to d_1, demands d_2 and d_3 will share an arc in every solution, and therefore, the pair (d_2, d_3) will be an element of the set L associated with the instance. This example shows that the set L is not derived solely from the structure of G, but also depends on $\bar{s}$ and D. If a higher value of $\bar{s}$ were allowed or the volume requirements of d_1, d_2, or d_3 were reduced, the paths associated with these demands might not share a common arc in every solution, and consequently, the pair (d_2, d_3) would not belong to L.

Finding the set $L(G, D, \bar{s})$ is a difficult task from a computational complexity point of view, as the following result shows.

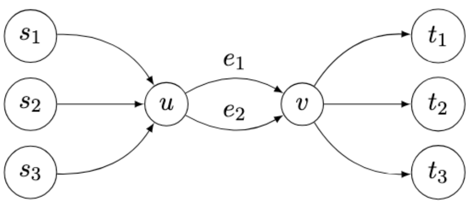

Fig. 1. Graph G associated with an example instance. The demand d_1 can use the arcs e_1 and e_2 and, due to the number of available slots ($\bar{s} = 4$), the demands d_2 and d_3 must share an arc, implying that the pair (d_2, d_3) is an element of the set L.

Theorem 2. *Let $d, d' \in D$, $d < d'$. Determining whether $(d, d') \in L(G, D, \bar{s})$ or not is NP-complete.*

In order to explore the dimension of $P(G, D, \bar{s})$, we need the following further definitions and results.

Definition 3. *For $d \in D$, we define the* all-paths polytope from $s(d)$ to $t(d)$ *to be*

$$PR(G,d) = \text{conv}\Big\{y \in \{0,1\}^{|E|} : \sum_{e\in\delta^-(u)} y_{de} - \sum_{e\in\delta^+(u)} y_{de} = \begin{cases} -1 & \text{if } u = s(d) \\ 1 & \text{if } u = t(d) \\ 0 & \text{otherwise} \end{cases} \quad \forall u \in V \Big\}.$$

The structure of the polytope $PR(G, d)$ is well understood, since the coefficient matrix is the incidence matrix of the graph G [1]. The following two lemmas associated with the dimension of this polyhedron present results that will be helpful in order to identify a minimal equation system of $P(G, D, \bar{s})$. For $d \in D$, we define the *feasible-demand graph associated to d* to be the graph $G_d := (V_d, E_d)$, where $E_d \subseteq E$ is the set of arcs used in the path associated to d in at least one feasible solution of $P(G, D, \bar{s})$ (in other words, $e \in E_d$ if and only if there exists a feasible solution of $P(G, D, \bar{s})$ such that the path assigned to d contains the arc e), and $V_d \subseteq V$ is the set of endpoints of the arcs in E_d.

Lemma 1. *For $d \in D$, we have* $\dim(PR(G_d, d)) = |E_d| - (|V_d| - 1)$.

Let $(y, l, x) \in P(G, D, \bar{s})$ be a feasible solution and let $d \in D$ be a demand. We define $y_{\neq d} \in \mathbb{R}^{(|D|-1)|E|}$ as the vector obtained from y by deleting the coordinates associated with d, that is, $y_{\neq d}$ is the projection of y on the variables $\{y_{d'e}\}_{d' \in D\setminus\{d\}, e\in E}$. If $A \subseteq P(G, D, \bar{s})$ and $d \in D$, we define $A_{\neq d} := \{y_{\neq d}$: there exists (l, x) such that $(y, l, x) \in A\}$.

Lemma 2. *Let F be a (not necessarily proper) face of $P(G, D, \bar{s})$, and assume $\alpha^\intercal y + \beta^\intercal l + \gamma^\intercal x = \pi_0$ for every $(y, l, x) \in F$. If*

(a) for each $d \in D$ there exists a set $A \subseteq F$ with $|E_d| - |V_d| + 2$ affinely independent points such that $|A_{\neq d}| = 1$, and
(b) there exists $\lambda_0 \in \mathbb{R}$ such that $\beta^\intercal l + \gamma^\intercal x = \lambda_0$ for every $(y, l, x) \in F$,

then α is a linear combination of the flow conservation constraints (2).

The hypothesis (a) in Lemma 2 is trivially satisfied if the number $\bar{s}$ of slots is large enough. We denote by $s_{\min}(G, D)$ the smallest integer s such that $P(G, D, s) \neq \emptyset$, i.e., $P(G, D, s_{\min}(G, D)) \neq \emptyset$ and $P(G, D, s_{\min}(G, D) - 1) = \emptyset$. We also define $\Delta_D := \max_{d\in D} v(d)$ to be the maximum volume over all demands. If, e.g., $\bar{s} \geq s_{\min}(G, D) + \Delta_D$, then the hypothesis (a) is guaranteed. On the other hand, the hypothesis (b) asks the face F not to be defined in terms of the y-variables, and this also holds if F is the polytope $P(G, D, \bar{s})$ itself.

Theorem 3. *If $\bar{s} \geq s_{\min}(G, D) + 2\Delta_D$, then constraints (2) together with*

$$x_{dd'} + x_{d'd} = 1 \qquad \forall (d, d') \in L(G, D, \bar{s}), \tag{8}$$

$$y_{de} = 0 \qquad \forall d \in D, e \in E \setminus E_d \tag{9}$$

define a minimal equation system for $P(G, D, \bar{s})$.

Corollary 1. *If $\bar{s} \geq s_{\min}(G, D) + 2\Delta_D$, then the dimension of $P(G, D, \bar{s})$ is $|D|(|E| - |V|) + |D|^2 + |D| - |L(G, D, \bar{s})| - \sum_{d\in D} |E \setminus E_d|$.*

Corollary (1) characterizes the dimension of $P(G, D, \bar{s})$ when $\bar{s} \geq s_{\min}(G, D) + 2\Delta_D$. If the number of available slots $\bar{s}$ is less than $s_{\min}(G, D)$, then there are no feasible solutions and therefore $P(G, D, \bar{s})$ is empty. The dimension of this polyhedron for $s_{\min}(G, D) \leq \bar{s} < s_{\min}(G, D) + 2\Delta_D$ is difficult to characterize, since it strongly depends on the structure of the graph G and the specifications of the demands. A similar situation holds for the classical vertex coloring polytope (when the number of available colors equals the chromatic number of the graph) and for interval coloring polytopes (with a similar hypothesis as the one presented here).

3. Valid inequalities

Although completely understanding the structure of $P(G, D, \bar{s})$ is a nontrivial task, in this section we start such an exploration by presenting families of valid inequalities and showing that, under certain conditions on the set of conflicting demands $L(G, D, \bar{s})$, these inequalities define facets of this polytope.

3.1. Relations to chromatic scheduling polytopes

When $L(G, D, \bar{s}) \neq \emptyset$, indicating that there exist pairs of demands sharing at least one arc in every solution, a proper conflict graph among the demands arises. The edges of this graph (corresponding to pairs of demands in $L(G, D, \bar{s})$) provide a close relation between RSA and interval coloring, and in this subsection we are interested in the polyhedral consequences of this fact.

Definition 4. *We define the* conflict graph *associated to $P(G, D, \bar{s})$ to be the weighted graph $G_C = (D, E_C, \mathbf{w})$, where $E_C = \{dd' \in D \times D : (d, d') \in L(G, D, \bar{s})$ or $(d', d) \in L(G, D, \bar{s})\}$, and $w_d = v(d)$ for every $d \in D$.*

As mentioned before, interval coloring involves assigning a set of consecutive colors to each vertex of a graph. Formally, given a weighted graph $G = (V, E, \mathbf{w})$ and a set $S = \{0, \ldots, \bar{s} - 1\}$ of colors, the goal is to assign an interval of w_i colors to the vertex i, for every $i \in V$, in such a way that no two adjacent vertices receive the same color. The classical integer programming formulation for interval coloring employs an integer variable $l_i \in \{0, \ldots, \bar{s} - 1\}$ for each $i \in V$, and a binary ordering variable x_{ij} for every $ij \in E$, $i < j$, in such a way that $x_{ij} = 1$ if and only if $l_i + w_i \leq l_j$. In this setting, a valid coloring is an assignment of values to these variables satisfying the following constraints.

$$l_i + w_i \leq l_j + \bar{s}(1 - x_{ij}) \qquad \forall i, j \in E, i < j, \tag{10}$$

$$l_j + w_j \leq l_i + \bar{s}x_{ij} \qquad \forall i, j \in E, i < j, \tag{11}$$

$$x_{ij} \in \{0, 1\} \qquad \forall i, j \in E, i < j, \tag{12}$$

$$l_i \in \{0, \ldots, \bar{s} - 1\} \qquad \forall i \in V. \tag{13}$$

Definition 5. *Given a weighted graph $G = (V, E, \mathbf{w})$ and a number $\bar{s} \in \mathbb{Z}_+$ of available colors, we define the* chromatic scheduling polytope *$C(G, \bar{s}) \subseteq \mathbb{R}^{|V|+|E|}$ to be the convex hull of all integer solutions $(l, x) \in \mathbb{R}^{|V|+|E|}$ satisfying constraints (10)-(13).*

Despite differences in their specific constraints and objectives, these problems share common structural polyhedral properties, the projection of $P(G, D, \bar{s})$ onto the l- and the x-variables being included in $C(G_C, \bar{s})$. This relation can be explored to derive helpful results. In particular, inequalities that are valid for $C(G_C, \bar{s})$ are also valid for $P(G, D, \bar{s})$, offering a way to use known results from chromatic scheduling polytopes to RSA, as follows.

Theorem 4. *If $\beta^\top l + \gamma^\top x \leq \pi_0$ is a valid inequality for $C(G_C, \bar{s})$, then the inequality $\beta^\top l + \gamma^\top x \leq \pi_0$ is also valid for $P(G, D, \bar{s})$.*

On the other hand, the following theorem presents necessary conditions for a facet-inducing inequality of $P(G, D, \bar{s})$ to define a facet of $C(G_C, \bar{s})$.

Theorem 5. *Let $\beta^\top l + \gamma^\top x \leq \pi_0$ be a facet-inducing inequality for $P(G, D, \bar{s})$ such that $\gamma_{dd'} = 0$ for every $(d, d') \in D \times D$ with $(d, d') \notin L(G, D, \bar{s})$ and $(d', d) \notin L(G, D, \bar{s})$. If $\bar{s} \geq s_{\min}(G, D) + 2\Delta_D$, then $\beta^\top l + \gamma^\top x \leq \pi_0$ induces a facet of $C(G_C, \bar{s})$.*

Theorem 5 provides hints on the structure of $P(G, D, \bar{s})$, by showing that facet-inducing inequalities of $P(G, D, \bar{s})$ can be transferred to $C(G_C, \bar{s})$ under specific conditions. Although this does not provide a characterization of all facet-inducing inequalities of $P(G, D, \bar{s})$, this result provides a necessary condition for an inequality in terms of the x- and l-variables to define a facet of $P(G, D, \bar{s})$ for an arbitrary value of $\bar{s}$. This provides a general strategy of searching for facet-inducing inequalities, namely taking families of facet-inducing inequalities for $C(G_C, \bar{s})$ and exploring their facetness properties for $P(G, D, \bar{s})$. This issue is explored in the next subsection.

3.2. Families of valid inequalities

We begin by focusing on constraints (4), which appear with a similar expression in chromatic scheduling polytopes and turn out to be strong for the DR-AOV formulation.

Theorem 6. *If $\bar{s} \geq s_{\min}(G, D) + 3\Delta_D$ and $L(G, D, \bar{s}) = \emptyset$, then the inequality (4) is facet-inducing for $P(G, D, \bar{s})$.*

Theorem 6 provides a facetness result for $L(G, D, \bar{s}) = \emptyset$, and it is interesting to explore whether this result can be extended to the case $L(G, D, \bar{s}) \neq \emptyset$. To this end, we introduce the following definitions, based on similar results for chromatic scheduling polytopes.

Definition 6. *Let $d \in D$ and let $K \subseteq N(d)$ be a clique in G_C. We define*

$$l_d \geq \sum_{k \in K} v(k) x_{kd} \tag{14}$$

to be the single clique inequality *associated with d and K.*

Figure 2 illustrates the assertion imposed by the single clique inequality. In this case, the cumulative volumes of the elements of the clique $K = \{k_1, k_2, k_3\}$, whose corresponding intervals are located to the left of the interval assigned to the demand d, induce a lower bound on the first slot assigned to the demand d.

This and the following families of inequalities to be presented, besides being inequalities that define facets of $C(G_c, \bar{s})$, satisfy the conditions given in Theorem 5, that is, the coefficients of the $x-$variables associated with the pairs of demands $(d, d') \in (D \times D) \setminus L(G, D, \bar{s})$ and all the $y-$variables are null in them. We will show that they are valid for $P(G, D, \bar{s})$ and, moreover, that a subset of them defines facets of $P(G, D, \bar{s})$.

Proposition 1. *The single clique inequality (14) is valid for $P(G, D, \bar{s})$.*

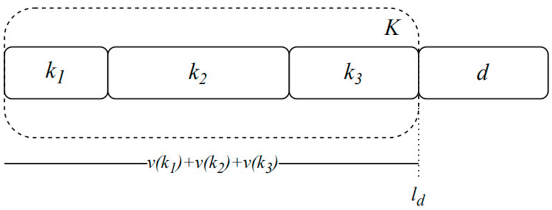

Fig. 2. The single clique inequality provides a lower bound on the first slot of demand d given by the elements of the clique $K = \{k_1, k_2, k_3\}$ located to the left of d.

We now explore conditions ensuring that (14) induces a facet of $P(G, D, \bar{s})$, based on similar arguments for chromatic scheduling polytopes [11]. If $G = (V, E, \mathbf{w})$ is a weighted graph and $i \in V$, we define $N(i) := \{j \in V : ij \in E\}$ to be the set of neighbors of i.

Definition 7. *Let $G = (V, E, \mathbf{w})$ be a weighted graph, and take $A \subseteq V$ and a clique $K \subseteq A$. We say that K* covers *A (or, alternatively, that K is a* covering clique *of A) if every node $k \in A \setminus K$ satisfies $w_k \leq \sum_{t \in K \setminus N(k)} w_t$.*

For every subset $A \subseteq V$ there exists a covering clique K of A, and such a clique can be found in polynomial time [11]. It turns out that being a covering clique for $N(d)$ is a sufficient condition for (14) to induce a facet, together with a lower bound on the number of available slots.

Theorem 7. *If K covers $N(d)$ in G_C and $\bar{s} \geq s_{\min}(G, D) + 3\Delta_D$, then the single clique inequality (14) is facet-inducing for $P(G, D, \bar{s})$.*

When $\bar{s} < s_{\min}(G, D) + 3\Delta_D$ we cannot guarantee that (14) is facet-inducing, although in this setting it is possible that this indeed holds, depending on the structure of the graph G and the specifications of the demands. The covering clique inequality (14) describes the relation between the first slot assigned to d and the first available slot in the interval $\{0, \ldots, \bar{s} - 1\}$. By resorting to Theorem 1 we can provide a symmetric inequality with the same properties.

Definition 8. *Let $d \in D$ and let $K \subseteq N(d)$ be a clique in G_C. We define*

$$l_d + v(d) \leq \bar{s} - \sum_{k \in K} v(k) x_{dk} \tag{15}$$

to be the single opposite clique inequality *associated with d and K.*

Corollary 2. *The single opposite clique inequality (15) is valid for $P(G, D, \bar{s})$. Moreover, if K covers $N(d)$ in G_C and $\bar{s} \geq s_{\min}(G, D) + 3\Delta_D$, then (15) is facet-inducing for this polytope.*

We now consider a family of valid inequalities describing the relation between the slot intervals assigned to two neighboring demands in G_C. Any such pair of demands always must be assigned non-overlapping intervals, giving rise to the following result. If $K \subseteq D$, we define $v(K) := \sum_{k \in K} v(k)$.

Definition 9. *Let $(d, d') \in L(G, D, \bar{s})$ and let $K \subseteq N(d) \cap N(d')$ be a clique in G_C. We define*

$$l_d + v(d) + \sum_{k \in K} v(k)(x_{dk} - x_{d'k}) \leq l_{d'} + (\bar{s} - v(K)) x_{d'd} \tag{16}$$

to be the double clique inequality *associated with d, d', and K.*

The double clique inequality (16) asserts that the last slot assigned to the demand d (namely, $l_d + v(d) - 1$) must be strictly smaller than the first slot assigned to the demand d' whenever $x_{dd'} = 1$. Furthermore, the distance between these two slots is lower-bounded by the volumes of all the demands in K whose slot intervals are assigned between the slot intervals of d and d'. Figure 3 provides an example of this situation, by enforcing a minimum distance between the last slot assigned to d' and the first slot assigned to d', in terms of the demands from the clique $K = \{k_1, k_2, k_3\}$ allocated between d and d'.

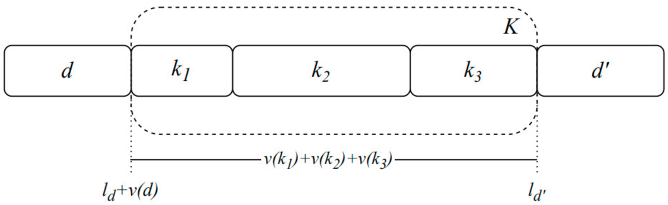

Fig. 3. The double clique inequality enforces a lower bound on the distance between the last slot assigned to d and the first slot assigned to d', whenever d is allocated before d'.

Proposition 2. *The double clique inequality (16) is valid for $P(G, D, \bar{s})$.*

Again, the double clique inequalities are facet-inducing if K is a covering clique for $N(d) \cap N(d')$ and there is a large enough number of available slots.

Theorem 8. *If K covers $N(d) \cap N(d')$ in G_C and $\bar{s} \geq s_{\min}(G, D) + 4\Delta_D$, then the double clique inequality (16) is facet-inducing for $P(G, D, \bar{s})$.*

4. Concluding remarks

In this work, we have started a polyhedral study of RSA by exploring the polytope associated with one of the most promising integer programming formulations for this problem. Besides identifying a minimal equation set for this polytope, we presented families of facet-inducing inequalities related to interval coloring based on a direct relation between this polytope and chromatic scheduling polytopes. This relation provides an interesting starting point for the exploration of the RSA polytope, since many facet-inducing inequalities of the former translate to the latter.

The results in Section 3 depend on knowing $L(G, D, \bar{s})$, a set that is computationally difficult to obtain, thus hindering the practical application of the results presented in this work. However, it is possible to focus on a subset of $L(G, D, \bar{s})$ in order to derive practical results that, while not capturing the full complexity of the original approach, may still provide meaningful insights. If we can reasonably compute a subset $L' \subseteq L(G, D, \bar{s})$, then the generated inequalities for L' are still valid, and this may provide an interesting contribution in practice. As an example, pairs of demands forced to use a bridge in the network are trivial members of L'. It would also be interesting to identify pairs of demands forced to use the same link in the network, even when the network has no bridges. These observations may allow to add nontrivial cuts for the DR-AOV formulation. This endeavor, together with the evaluation of the practical contribution of such an implementation, is left as future work.

References

[1] Ahuja, R.K., Magnanti, T.L., Orlin, J.B., 1993. Network Flows: Theory, Algorithms, and Applications. Prentice hall.

[2] Bertero, F., Bianchetti, M., Marenco, J., 2018. Integer programming models for the routing and spectrum allocation problem. TOP 26, 465–488.

[3] Christodoulopoulos, K., Tomkos, I., Varvarigos, E.A., 2011. Elastic bandwidth allocation in flexible ofdm-based optical networks. Journal of Lightwave Technology 29, 1354–1366.

[4] Colares, R., Kerivin, H., Wagler, A., 2022. An extended formulation for the constrained routing and spectrum assignment problem in elastic optical networks, in: Joint ALIO/EURO International Conference 2021-2022 on Applied Combinatorial Optimization, pp. 5–10.

[5] Gerhardt, A., 1999. Polyedrische Untersuchungen von Zwei-Maschinen-Scheduling-Problemen mit Antiparallelitätsbedingungen. Master thesis. Technische Universität Berlin.

[6] Hadhbi, Y., Kerivin, H., Wagler, A., 2019. A novel integer linear programming model for routing and spectrum assignment in optical networks, in: 2019 Federated Conference on Computer Science and Information Systems (FedCSIS), IEEE. pp. 127–134.

[7] Klinkowski, M., Pedro, J., Careglio, D., Pióro, M., Pires, J., Monteiro, P., Solé-Pareta, J., 2010. An overview of routing methods in optical burst switching networks. Optical Switching and Networking 7, 41–53.

[8] Klinkowski, M., Walkowiak, K., 2011. Routing and spectrum assignment in spectrum sliced elastic optical path network. IEEE Communications Letters 15, 884–886.

[9] Kubale, M., 1989. Interval vertex-coloring of a graph with forbidden colors. Discrete Math. 74, 125–136.

[10] Marenco, J., Wagler, A., 2006. On the combinatorial structure of chromatic scheduling polytopes. Discrete Applied Mathematics 154, 1865–1876. Traces of the Latin American Conference on Combinatorics, Graphs and Applications.

[11] Marenco, J., Wagler, A., 2007. Chromatic scheduling polytopes coming from the bandwidth allocation problem in point-to-multipoint radio access systems. Annals of Operations Research 150, 159–175.

[12] Marenco, J., Wagler, A., 2009. Facet-inducing inequalities for chromatic scheduling polytopes based on covering cliques. Discret. Optim. 6, 64–78.

[13] Velasco, L., Klinkowski, M., Ruiz, M., Comellas, J., 2012. Modeling the routing and spectrum allocation problem for flexgrid optical networks. Photonic Network Communications 24, 177–186.

[14] Wang, Y., Cao, X., Hu, Q., , Pan, Y., 2012. Towards elastic and fine-granular bandwidth allocation in spectrum-sliced optical networks. J. Opt. Commun. Netw. 4, 906–917.

[15] Żotkiewicz, M., Pióro, M., Ruiz, M., Klinkowski, M., Velasco, L., 2013. Optimization models for flexgrid elastic optical networks, in: 2013 15th International Conference on Transparent Optical Networks (ICTON), IEEE. pp. 1–4.