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A polyhedral study of the maximum-impact coloring problem on hypergraphs

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Abstract

Given a graph $G = (V, E)$ and a hypergraph $H = (V, \mathcal{F})$ over the same set of vertices, and a finite color set C, the *maximum-impact coloring problem on hypergraphs* asks for a C-coloring of G maximizing the number of hyperedges of H whose vertices are assigned the same color. This problem arises in the context of classroom assignment to courses, in which we need to assign a classroom to each lecture and we wish to assign the same classroom to all lectures from the same course. Since imposing this last concern as a constraint may be too restrictive, we seek to maximize the number of courses such that all of its lectures are assigned to the same classroom. In this work we present an integer programming formulation for this NP-hard problem and we explore the associated polytope. We present three families of facet-inducing inequalities, and we report computational experiments suggesting that the dynamic addition of these inequalities within a branch and cut environment may be effective in practice.

Keywords: classroom allocation, integer programming, polyhedral combinatorics

1. Introduction

Classroom allocation is a classical combinatorial optimization problem, which consists in assigning a classroom to each lecture to be given in such a way that overlapping lectures are assigned different classrooms. This setting is neatly modeled as a graph coloring problem, in which vertices represent lectures, edges join pairs of overlapping lectures, and the set of available colors represents classrooms. When no additional constraints are present, classroom allocation problems can be tackled with integer programming techniques quite successfully but, if additional considerations are to be taken into account [5, 6], then these problems can quickly become intractable and sophisticated techniques must be resorted to in order to address these problems in practice [7, 8, 9].

In this work we are interested in a particular classroom allocation model arising from a practical application. We are given the timetable of lectures that will take place through a typical week. These lectures have already been scheduled, so each lecture has an associated weekday, a starting time, and an ending time. Also, each lecture corresponds to a course, in such a way that each course is composed by one or more lectures throughout the week. We also have a set of available classrooms, and we assume that each classroom can host any lecture. The problem consists in assigning one classroom to each lecture in such a way that overlapping lectures do not receive the same classroom, maximizing the number of courses all of whose lectures are assigned to the same classroom. In other words, we would like all lectures from the same course to be allocated to a single classroom, although this is not a strong constraint but an objective to be maximized.

We can represent the lectures with a graph $G = (V, E)$, in such a way that V is the set of lectures and E contains an edge for each pair of overlapping lectures. We can represent courses with a hypergraph $H = (V, \mathcal{F})$ over the same set V of vertices, in such a way that each hyperedge in $\mathcal{F}$ represents a course and is given by the set of vertices associated with the course lectures. Finally, we are given a set C of classrooms. In this setting, a C-coloring of G (i.e., a function $c : V \to C$ such that $c(i) \neq c(j)$ whenever $ij \in E$) provides a feasible allocation of classrooms to lectures.

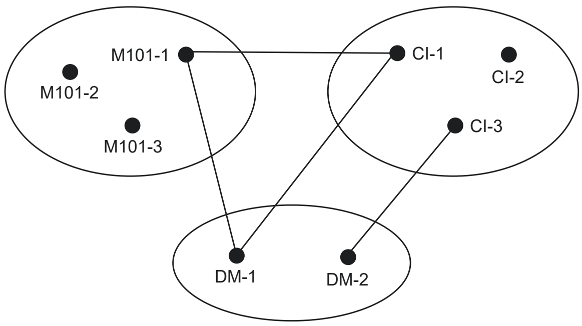
Definition 1. *Given a graph $G = (V, E)$ and a hypergraph $H = (V, \mathcal{F})$ over the same set of vertices, and a finite color set C, the* maximum-impact coloring problem on hypergraphs *asks for a C-coloring c of G maximizing $|\{f \in \mathcal{F} : c(i) = c(j) \text{ for every } i, j \in f\}|$, i.e., the number of hyperedges of H all of whose vertices are assigned the same color.*

Figure 1 provides an example instance of the classroom allocation problem, and the graph/hypergraph associated with this instance. Each course is associated to a hyperedge in H, whereas each pair of overlapping lectures generates an edge in G.

When every hyperedge $f \in \mathcal{F}$ has $|f| = 2$, i.e., when H is a graph, then the problem reduces to the *maximum-impact coloring problem in graphs* [1]. In the maximum-impact coloring problem in graphs, we have an edge in H between each pair of lectures corresponding to the same course, and the objective is to maximize the number of edges of H whose endpoints are

Course	Lecture	Lecture ID	Day	Starting time	Ending time
Math 101	Theoretical exposition	M101-1	Monday	10:00	12:00
	Theoretical exposition	M101-2	Wednesday	10:00	12:00
	Problem-solving session	M101-3	Friday	14:00	16:00
Calculus I	Theoretical exposition	CI-1	Monday	11:00	13:00
	Problem-solving session	CI-2	Wednesday	14:00	16:00
	Exercise class	CI-3	Friday	10:00	12:00
Discrete Math	Theoretical exposition	DM-1	Monday	11:00	13:00
	Problem-solving session	DM-2	Friday	9:00	11:00

(a)



(b)

Figure 1: (a) Example instance of the classroom allocation problem, and (b) its associated graph $G = (V, E)$ and hypergraph $H = (V, \mathcal{F})$ (solid lines represent edges in E, and ellipses represent hyperedges in $\mathcal{F}$). If, e.g., all three lectures from the course "Math 101" are allocated to the classroom 1, all three lectures from the course "Calculus I" are allocated to the classroom 2, the lecture DM-1 is allocated to the classroom 3, and the lecture DM-2 is allocated to the classroom 4, then the associated solution of the maximum-impact coloring problem on hypergraphs has objective function equal to 2 (given by the first two courses).

assigned the same color. This corresponds to collecting a one-unit prize whenever two lectures from the same course receive the same classroom. In this setting, a three-lecture course will form a triangle in H and will contribute three units to the objective function if all its lectures are assigned the same classroom. Similarly, a four-lecture course will contribute six units to the objective function if all its lectures are assigned the same classroom. This way of measuring the objective function may penalize courses with few lectures in favor or larger courses, which contribute larger amounts to the objective function when all its lectures get the same classroom. In contrast, in the maximum-impact coloring problem on hypergraphs considered in this work, a course always contributes one unit to the objective function when all its lectures receive the same classroom. This may be a fairer measure of the quality of the classroom assignment, since the objective function expresses the number of courses with an ideal classroom allocation, and this motivates our interest in this problem.

The maximum-impact coloring problem in graphs was first studied in [1], in which several facet-inducing inequalities were identified and a branch and cut procedure was proposed. In the follow-up work [2], some of the previous facetness results were unified within a general framework that allowed to explain these results as corollaries of a more general theorem. In particular, two facet-preserving procedures were presented for the maximum-impact coloring problem in graphs, and the application of these procedures to simple valid inequalities allowed to recover many of the facetness results in [1].

The maximum-impact coloring problem in graphs is NP-hard even if G is an interval graph and H is the disjoint union of cliques [1]. This implies that the maximum-impact coloring problem on hypergraphs is also NP-hard and, due to this fact, we propose in this work to employ integer programming techniques in order to solve this problem with optimality. To the best of our knowledge, the maximum impact coloring problem on hypergraphs has not been addressed previously in the optimization literature.

The remainder of this work is organized as follows. Section 2 presents an integer programming formulation for the maximum-impact coloring problem on hypergraphs and defines the associated polytope. Section 3 identifies three families of valid inequalities and explores conditions ensuring that these inequalities induce facets of the polytope we are interested in. Section 4 reports our computational experiments with these ideas, and Section 5 closes the paper with concluding remarks and lines for future research.

2. Integer programming formulation

An integer programming formulation for the maximum-impact coloring problem on hypergraphs can be given by employing similar ideas to the ones presented in [1]. For each vertex $i \in V$ and each color $c \in C$, we introduce the binary variable x_{ic} in such a way that $x_{ic} = 1$ if the vertex i is assigned the color c, and $x_{ic} = 0$ otherwise. For each $f \in \mathcal{F}$, we introduce the binary variable z_f in such a way that $z_f = 1$ only if $x_{ic} = x_{jc}$ for every $i, j \in f$ and for some $c \in C$ (i.e., $z_f = 1$ only if all the vertices from f are assigned the same color). In this setting, the maximum-impact coloring problem on hypergraphs can be formulated as follows.

$$\max \quad \sum_{f \in \mathcal{F}} z_f \tag{1}$$

$$\sum_{c \in C} x_{ic} = 1 \qquad \forall i \in V \tag{2}$$

$$x_{ic} + x_{jc} \leq 1 \qquad \forall ij \in E,\ \forall c \in C \tag{3}$$

$$z_f \leq 1 + x_{ic} - x_{jc} \qquad \forall f \in \mathcal{F},\ \forall i, j \in f,\ \forall c \in C \tag{4}$$

$$x_{ic} \in \{0, 1\} \qquad \forall i \in V,\ \forall c \in C \tag{5}$$

$$z_f \in \{0, 1\} \qquad \forall f \in \mathcal{F} \tag{6}$$

The objective function (1) asks to maximize the number of hyperedges from H all of whose constituent vertices are assigned the same color. Constraints (2) ensure that every vertex is assigned exactly one color, whereas constraints (3) impose that adjacent vertices must be assigned different colors. Constraints (4) specify that if the hyperedge $f \in \mathcal{F}$ has two vertices that are assigned different colors, then $z_f = 0$. We do not impose constraints asking $z_f = 1$ when all the vertices from f receive the same color, since in any optimal solution this will be the case. Finally, constraints (5)-(6) specify the domains for the variables.

The formulation (1)-(6) suffers from symmetry issues, since any relabeling of the colors provides a new (albeit equivalent) feasible solution within the model. This situation may not be an issue if additional constraints are present as, e.g., constraints limiting which classroom can be assigned to each lecture (e.g., due to capacity considerations), but it can be problematic for a general integer programming solver if this implies that the solver has to explore a large number of equivalent feasible solutions. In order to evaluate a formulation that avoids these issues, we performed preliminary experiments with a formulation based on the representatives formulation for the classical vertex coloring problem [3]. Such a formulation uses a binary variable y_{ji}

for $i \in V$ and $j \in \overline{N}(i) := \{k \in V : ik \notin E\}$ in such a way that $y_{ji} = 1$ if i and j are assigned the same color and j is the *representative* of their color class. Unfortunately, according to our experiments this formulation has a poor computational performance, so we stick to the formulation (1)-(6) in this work.

Definition 2. *We define $\mathcal{P}_{HMIC}(G, H, C) \subset \mathbb{R}^{|V||C|+|\mathcal{F}|}$ to be the convex hull of all points $(x, z) \in \mathbb{R}^{|V||C|+|\mathcal{F}|}$ satisfying constraints (2)-(6).*

Throughout this work we assume hypotheses on the input data, which are usual in the context of classroom allocation to lectures. We state these assumptions formally for future reference.

Assumption 1. *We assume that $ij \notin E$ for every $f \in \mathcal{F}$ and every $i, j \in f$ (i.e., no two lectures from the same course overlap).*

If Assumption 1 does not hold, i.e., there exists an edge $ij \in E$ between two vertices $i, j \in f$ for some $f \in \mathcal{F}$, then the variable z_f cannot take value 1 in any feasible solution, so we can set $z_f = 0$ in this case.

Assumption 2. *We assume that $f \cap g = \emptyset$ for every $f, g \in \mathcal{F}$, $f \neq g$ (i.e., no lecture belongs to more than one course).*

Assumptions 1 and 2 will be necessary for characterizing the dimension of $\mathcal{P}_{HMIC}(G, H, C)$ and in the facetness proofs. None of the validity proofs contained in this work rely on these assumptions. We now explore the dimension of $\mathcal{P}_{HMIC}(G, H, C)$. For $i \in V$ and $c \in C$, we denote by $e_{ic} \in \{0, 1\}^{|V||C|}$ the unit vector associated with the variable x_{ic}. For $f \in \mathcal{F}$, we define $e_f \in \{0, 1\}^{|\mathcal{F}|}$ to be the unit vector associated with the variable z_f. For $\lambda \in \mathbb{R}^{|V||C|}$ and $i \in V$, we call $\lambda_i \in \mathbb{R}^{|C|}$ the vector $\lambda_i = (\lambda_{i1}, \ldots, \lambda_{i|C|})^\intercal$. We denote by $\hat{0}$ the all-zeros vector with the appropriate dimension. Finally, let us denote by $\chi(G)$ the chromatic number of G, i.e., the smallest number of colors needed to color the graph G.

Proposition 1. *If $\chi(G) < |C|$, then $\mathcal{P}_{HMIC}(G, H, C)$ has dimension $|V|(|C| - 1) + |\mathcal{F}|$.*

Proof. Let $\lambda \in \mathbb{R}^{|V||C|}$, $\mu \in \mathbb{R}^{|\mathcal{F}|}$, and $\lambda_0 \in \mathbb{R}$ such that every $(x, z) \in \mathcal{P}_{HMIC}(G, H, C)$ satisfies $\lambda^\intercal x + \mu^\intercal z = \lambda_0$. We shall show that (λ, μ) is a combination of the coefficient vectors of the model constraints (2), thus settling the result.

Claim 1: $\mu_f = 0\ \forall f \in \mathcal{F}$. Take any feasible solution using $\chi(G)$ colors. Since $|C| > \chi(G)$, there exists a color, say c, not assigned to any vertex. Construct a new coloring from this feasible solution by assigning color c to all vertices in f (which is feasible by Assumption 1), and let $x \in \{0,1\}^{|V||C|}$ represent this coloring. Consider the solutions $w^1 = (x, \hat{0})$ and $w^2 = (x, e_f)$. Since w^1 and w^2 are feasible solutions only differing in the z_f-variable, then $(\lambda, \mu)^\intercal w^1 = \lambda_0 = (\lambda, \mu)^\intercal w^2$ implies $\mu_f = 0$. $\lozenge$

Claim 2: $\lambda_{vc} = \lambda_{vc'}\ \forall c, c' \in C,\ v \in V$. Take any feasible solution using $\chi(G)$ colors and such that the color c' is not used in any vertex. Exchange the vertex colors in such a way that v is assigned the color c, and let $x^1 \in \{0,1\}^{|V||C|}$ represent this coloring. Construct x^2 from x^1 by setting $x^2_{vc'} = 1$, $x^2_{vc} = 0$, and leaving the remaining variables unchanged. Again, the points $w^1 = (x^1, \hat{0})$ and $w^2 = (x^2, \hat{0})$ are feasible solutions, and $(\lambda, \mu)^\intercal w^1 = \lambda_0 = (\lambda, \mu)^\intercal w^2$ implies $\lambda_{vc} = \lambda_{vc'}$. $\lozenge$

This shows that $\mu = \hat{0}$ and that λ is a linear combination of the coefficient vectors of the model constraints (2). Since these vectors are linearly independent, the result follows. $\square$

When $\chi(G) < |C|$, Proposition 1 states that the only equations satisfied by all the points in $\mathcal{P}_{HMIC}(G, H, C)$ are linear combinations of the model constraints (2) asking each vertex to be assigned exactly one color. This implies that constraints (2) define a minimal equation system for $\mathcal{P}_{HMIC}(G, H, C)$, hence provide a maximal set of non-redundant hyperplanes including this polytope. A similar result holds for the classical vertex coloring problem. The formulation considered in this work also contains the z-variables, but since any solution with $z_f = 1$ for some $f \in \mathcal{F}$ can be modified to another feasible solution by setting $z_f = 0$, then these variables do not add additional equations to the analysis of the dimension. When $\chi(G) < |C|$, Proposition 1 provides a key starting point for determining whether the valid inequalities to be identified in Section 3 define facets of the associated polytope.

The dimension of $\mathcal{P}_{HMIC}(G, H, C)$ for $\chi(G) = |C|$ seems difficult to characterize, since it may depend on the particular structure of G. This situation also holds for the classical vertex coloring problem and, to the best of our knowledge, has not been tackled in the literature so far. It would be interesting to characterize –at least partially– the dimension of coloring polytopes coming from the standard formulation when $\chi(G) = |C|$, in order to bridge this theoretical gap in polyhedral combinatorics applied to graph coloring.

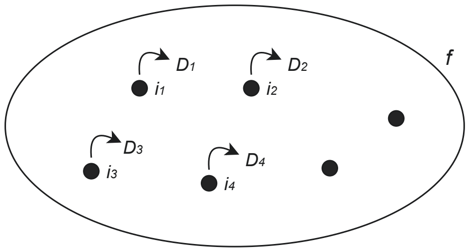


Figure 2: Example support of the partition inequality (7) for $n = 4$. The ellipse represents the hyperedge f in H and, for $j = 1, \ldots, n$, the curved arrow associating the vertex i_j to the color set D_j represents that the variables in $\{x_{i_j d}\}_{d \in D_j}$ are present in the inequality.

3. Valid inequalities

In this section we explore three families of valid inequalities for the polytope $\mathcal{P}_{HMIC}(G, H, C)$, and we identify conditions ensuring facetness for them.

3.1. Partition inequalities

In this sub-section we present a family of valid inequalities that generalizes a similar familiy of valid inequalities for the maximum-impact coloring polytope (on graphs).

Definition 3. *Fix $f \in \mathcal{F}$ and let $i_1, \ldots, i_n \in f$, with $n \geq 2$ and $i_j \neq i_k$ for $j, k = 1, \ldots, n$, $j \neq k$. Let $D_1, \ldots, D_n \subseteq C$ be nonempty and pairwise disjoint sets of colors such that $\cup_{j=1}^{n} D_j = C$. In this setting, we define*

$$z_f \leq \sum_{j=1}^{n} \sum_{d \in D_j} x_{i_j d} \tag{7}$$

to be the partition inequality *associated with $i_1, \ldots, i_n$ and $D_1, \ldots, D_n$.*

Figure 2 presents an example support of the partition inequalities. This inequality asserts that z_f must take null value if the vertex i_j receives a color in $C \backslash D_j$, for $j = 1, \ldots, n$, as the first part of the proof of the following result shows.

Theorem 1. *The partition inequality (7) is valid for $\mathcal{P}_{HMIC}(G, H, C)$. Moreover, if $\chi(G) + 1 < |C|$ then (7) induces a facet of $\mathcal{P}_{HMIC}(G, H, C)$.*

Proof. Assume that $(x, z) \in \mathcal{P}_{HMIC}(G, H, C)$ is an integer feasible solution that does not satisfy (7). This implies that $z_f = 1$ and $\sum_{d \in D_j} x_{i_j d} = 0$ for $j = 1, \dots, n$. Therefore, i_j is assigned a color from $C \backslash D_j$ for each $j = 1, \dots, n$. We claim that there exist at least two vertices in $I = \{i_1, \dots, i_n\}$ receiving different colors. If this were not the case then all vertices in I would receive the same color $c \in C$ and, since $\{D_1, \dots, D_n\}$ is a partition of C, there exists some $k \in \{1, \dots, n\}$ such that $c \in D_k$, implying $\sum_{d \in D_j} x_{i_j d} \geq 1$, a contradiction with the initial assumption. Thus, there must be at least two vertices in $I \subseteq f$ with different colors, which contradicts the assumption that $z_f = 1$. Since (x, z) is an arbitrary feasible solution, we conclude that (7) is a valid inequality.

Now for facetness. To this end, let F be the face of $\mathcal{P}_{HMIC}(G, H, C)$ induced by (7), and suppose $\lambda^\intercal x + \mu^\intercal z = \lambda_0$ for every $(x, z) \in F$. We shall show that (λ, μ) is a linear combination of the model constraints (2) and the coefficient vector of (7), thus showing that F is a facet of $\mathcal{P}_{HMIC}(G, H, C)$.
Claim 1: $\boldsymbol{\lambda_{kc} = \lambda_{kc'}}$ $\forall \boldsymbol{k \notin f}$, $\forall \boldsymbol{c, c' \in C}$. Let $c'' \in C \backslash \{c, c'\}$. Take any feasible solution $w^0 = (x^0, \hat{0})$ using exactly $\chi(G)$ colors, such that no vertex is assigned the colors c' and c'', and such that the vertex k is assigned the color c (such a solution can be obtained from an optimal coloring by renaming the colors). Construct the solution $w^1 = (x^1, e_f)$ from w^0 by assigning color c'' to every vertex in f, and leaving the remaining variables unchanged. Since $c'' \in D_j$ for some $j = \{1, \dots, n\}$, we have $\sum_{d \in D_j} x_{i_j d} = 1$, hence $\sum_{j=1}^{n} \sum_{d_j \in D_j} x_{i_j d_j} = 1$ and w^1 satisfies (7) with equality. Construct a new solution $w^2 = (x^2, e_f)$ from w^1 by assigning color c' to the vertex k and leaving the remaining variables unchanged. This modification to w^1 is possible since w^1 does not use the color c'. This new solution also satisfies (7) with equality, hence $\lambda^\intercal x^2 + \mu^\intercal e_f = \lambda^\intercal x^1 + \mu^\intercal e_f$, implying $\lambda_{kc'} = \lambda_{kc}$. $\diamondsuit$
Claim 2: $\boldsymbol{\mu_{f'} = 0}$ $\forall \boldsymbol{f' \in \mathcal{F}}$, $\boldsymbol{f' \neq f}$. Let $w = (x^1, \hat{0})$ be a feasible solution using $\chi(G) + 1$ colors, such that all the vertices in f' receive the same color. Since $\chi(G) + 1 < |C|$, such a solution can be constructed from an optimal coloring by setting an unused color to all vertices from f'. Construct $\overline{w} = (\overline{x}, \overline{z})$ as follows.

- If all the vertices in f are assigned the same color in w, then take $\overline{x} = x^1$ and $\overline{z} = e_f$.
- Otherwise, take an unused color in x^1 (such a color exists since $|C| > \chi(G) + 1$), and assign this color to all vertices in f. By Assumption 2, this construction does not modify the color assigned to the vertices in f'. Let $\overline{x}$ represent the obtained coloring, and set $\overline{z} = e_f$.

Finally, construct the solution $\overline{w}' = (\overline{x}, e_f + e_{f'})$. Again, it is not difficult to verify that $\overline{w}$ and $\overline{w}'$ satisfy (7) with equality, implying $\mu_{f'} = 0$. $\Diamond$

Claim 3: $\boldsymbol{\mu_f = \lambda_{i_j d_m} - \lambda_{i_j d_j}}$ $\forall \boldsymbol{j, m \in \{1, \ldots, n\}}$**,** $\boldsymbol{j \neq m}$**,** $\forall \boldsymbol{d_j \in D_j}$**,** $\forall \boldsymbol{d_m \in D_m}$**.** Consider an optimal coloring x such that the colors d_m and d_j are not assigned to any vertex (such a solution exists since $|C| > \chi(G) + 1$). Construct $w^1 = (x^1, e_f)$ from x by setting the color d_j to all vertices in f and leaving the remaining variables unchanged. Construct also the solution $w^2 = (x^2, \hat{0})$ from w^1 by assigning color d_m to the vertex i_j and leaving the remaining variables unchanged. Again, we have $w^1, w^2 \in F$, implying $\lambda_{i_j d_j} + \mu_f = \lambda_{i_j d_m}$. $\Diamond$

Claim 4: $\boldsymbol{\lambda_{kd_j} = \lambda_{kd_m}}$ $\forall \boldsymbol{k \in f \backslash \{i_1, i_2, \ldots, i_n\}}$**,** $\forall \boldsymbol{d_j \in D_j}$**,** $\forall \boldsymbol{d_m \in D_m}$**,** $\boldsymbol{m \neq j}$**.** Let $w = (x, \hat{0})$ be a feasible solution with $\chi(G)$ colors such that the colors d_j and d_m are not assigned to any vertex (again, such a solution exists since $|C| > \chi(G)+1$). Take $w^1 = (x^1, e_f)$ to be the feasible solution obtained from w by assigning the color d_j to all vertices in f and leaving the remaining variables unchanged. Construct $w^2 = (w^2, \hat{0})$ from w^1 by assigning color d_m to the vertices k and i_j (which are not adjacent by Assumption 1), i.e., $x^2_{kd_m} = x^2_{i_j d_m} = 1$. We have that w^2 satisfies (7) since all vertices from f with the exception of i_j are assigned the color d_j, and i_j receives some color $d_m \notin D_j$, hence $z_f = 0 = \sum_{j=1}^{n} \sum_{d_s \in D_s} x_{i_s d_s}$. Since $w^1, w^2 \in F$, we have $\mu_f + \lambda_{kd_j} + \lambda_{i_j d_j} = \lambda_{kd_m} + \lambda_{i_j d_m}$ which, together with Claim 3, implies $\lambda_{kd_j} = \lambda_{kd_m}$. $\Diamond$

Claim 5: $\boldsymbol{\lambda_{kd_j} = \lambda_{kd'_j}}$ $\forall \boldsymbol{k \in f \setminus \{i_1, i_2, \ldots, i_n\}}$**,** $\forall \boldsymbol{d_j, d'_j \in D_j}$**.** This claim follows directly from Claim 4. $\Diamond$

Claim 6: $\boldsymbol{\lambda_{i_j d_m} = \lambda_{i_j d'_m}}$ $\forall$ $\boldsymbol{d_m, d'_m \in D_m}$**,** $\forall \boldsymbol{m, j \in \{1, \ldots, n\}}$**,** $\boldsymbol{m \neq j}$**.** Claim 3 implies $\mu_f = -\lambda_{i_j d_j} + \lambda_{i_j d_m}$ and, similarly, $\mu_f = -\lambda_{i_j d'_j} + \lambda_{i_j d_m}$. This implies $\lambda_{i_j d_j} = \lambda_{i_j d'_j}$ for every $d_j, d'_j \in D_j$. $\Diamond$

These claims imply that (λ, μ) is a linear combination of the model constraints (2) and the coefficient vector of (7), hence F is a facet of $\mathcal{P}_{HMIC}(G, H, C)$. $\square$

Given a fractional solution $w^* = (x^*, z^*) \in \mathbb{R}^{|V||C|+|\mathcal{F}|}$ in the linear relaxation of the formulation (2)-(6), the *separation problem* associated with the partition inequalities consists in determining whether any partition inequality is violated by w^*. When n is fixed, the separation problem associated with the partition inequalities can be solved in polynomial time, by considering all hyperedges $f \in \mathcal{F}$, considering all subsets $\{i_1, \ldots, i_n\}$ of f with n vertices, and assigning each color $d \in C$ to D_{j_d}, where $j_d = \text{argmin}\{x_{i_j d} : j = 1, \ldots, n\}$. This way, the contribution of w^* to the right-hand side of the partition inequality associated with f and $\{i_i, \ldots, i_n\}$ is minimized. Therefore, the separation problem has a positive outcome if and only if there exists $f \in \mathcal{F}$ and $\{i_1, \ldots, i_n\} \subseteq f$ for which the inequality constructed by this procedure is violated. We resort to this fact in our implementation (to be described in Section 4, where further details are provided) by applying this procedure for $n = 2$ and $n = |f|$. Although this argument shows that this separation problem is fixed-parameter tractable (for the parameter n), it would be interesting to explore whether the complete separation problem can be solved in polynomial time.

3.2. Union of cliques inequalities

Let $f \in \mathcal{F}$ and let $v, w \in f$. Assume there exists a vertex $j \in V \setminus f$ with $jv \in E$ and $jw \notin E$. In this situation, if j and w are assigned the same color, then v must receive a different color, thus implying $z_f = 0$. The following family of inequalities captures this fact in a more general setting (see Figure 3).

Definition 4. *Let $f \in \mathcal{F}$ and let $K \subseteq V$ be a nonempty clique in G such that $f \cap K = \{v\}$, and consider a vertex $w \in f \setminus \{v\}$. Let also $K' \subseteq V$ be a (possibly empty) clique in G such that*

(a) if $k' \in K'$ then $k'k \in E$ for every $k \in K \setminus \{v\}$,
(b) if $k' \in K'$ then $k'v \notin E$,
(c) if $k' \in K'$ then $k'v' \in E$ for some $v' \in f$.

Finally, fix a color $c \in C$. We define

$$z_f + x_{wc} + \sum_{k \in K \setminus f} x_{kc} + \sum_{k \in K'} x_{kc} \leq 2 \tag{8}$$

to be the union of cliques inequality *associated with v, w, K, and K'.*

Theorem 2. *The union of cliques inequality (8) is valid for $\mathcal{P}_{HMIC}(G, H, C)$. Moreover, if*

1. *w contains no neighbors in K,*
2. *$K' \cup K \setminus \{v\}$ is a maximal clique,*
3. *for every $u \in f \setminus \{v\}$ there exists $k \in K \setminus \{v\}$ such that $uk \notin E$,*
4. *there exists $k' \in K'$ such that $wk' \notin E$,*
5. *$K \setminus \{v\} \neq \emptyset$, and*
6. *$\chi(G) + 1 < |C|$,*

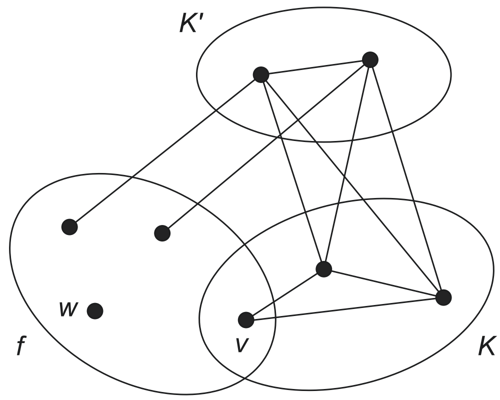


Figure 3: Example of the structure supporting the union of cliques inequality (8) for $|K| = 3$ and $|K'| = 2$. Note that there are not edges between vertices in f, and that there are no edges between w and the vertices in K'.

then (8) induces a facet of $\mathcal{P}_{HMIC}(G, H, C)$.

Proof. Let $(x, z) \in \mathcal{P}_{HMIC}(G, H, C)$ be a feasible solution and suppose that (x, z) violates (8), i.e.,

$$z_f + x_{wc} + \sum_{k \in K \setminus f} x_{kc} + \sum_{k \in K'} x_{kc} \geq 3. \tag{9}$$

The left-hand-side of (9) contains four terms, each of which can take value at most 1, so $\sum_{k \in K \setminus f} x_{kc} + \sum_{k \in K'} x_{kc} \geq 1$. Consider the following cases.

1. If $\sum_{k \in K \setminus \{v\}} x_{kc} = 1$ then no vertex from K' can get the color c, hence $\sum_{k \in K'} x_{kc} = 0$. In order to satisfy (9) we must have $x_{wc} = z_f = 1$, but this implies $x_{vc} = 1$, a contradiction since $v \in K$.
2. If $\sum_{k \in K'} x_{kc} = 1$, then no vertex from $K \setminus \{v\}$ can get the color c, since these vertices are adjacent to all vertices in K'. Therefore, in order to satisfy (9) we must have $x_{wc} = z_f = 1$, hence all vertices in f must get color c, contradicting the hypothesis (c) on K'.

Since in both cases we arrive at a contradiction, we conclude that (x, z) cannot violate (8). Since (x, z) is an arbitrary feasible solution, the validity of (8) follows.

Now for facetness. To this end, let F be the face of $\mathcal{P}_{HMIC}(G, H, C)$ induced by (8), and suppose $\lambda^\intercal x + \mu^\intercal z = \lambda_0$ for every $(x, z) \in F$. We shall

show that (λ, μ) is a linear combination of the model constraints (2) and the coefficient vector of (8), thus showing that F is a facet of $\mathcal{P}_{HMIC}(G, H, C)$.
Claim 1: $\boldsymbol{\mu_{f'} = 0,\ \forall f' \in \mathcal{F} \setminus \{f\}}$**.** Let $d \in C\backslash\{c\}$. Take any optimal coloring such that no vertex is assigned the color c and the color d (this is feasible since $\chi(G)+1 < |C|$). Construct a new coloring by setting the color c to every vertex in f and by setting the color d to every vertex in f' (this is possible by Assumption 2) and let $x \in \{0,1\}^{|V||C|}$ represent the obtained coloring. Define $w^1 = (x, e_f + e_{f'})$ and note that $w^1 \in F$. Also consider the solution $w^2 = (x, e_f)$, which also belongs to F. Since $w^1, w^2 \in F$, we have $(\lambda, \mu)^\intercal w^1 = \lambda_0 = (\lambda, \mu)^\intercal w^2$, implying $\mu_{f'} = 0$. $\diamondsuit$
Claim 2: $\boldsymbol{\lambda_{v'c} = \lambda_{v'd'}\ \forall d' \in C\backslash\{c\},\ \forall v' \notin f \cup K \cup K'}$**.** Let $d \in C\backslash\{c, d'\}$. Take any optimal coloring such that no vertex is assigned the color c nor the color d, and rename the colors in such a way that v' is assigned the color d'. Consider the following cases.

- If v' has no neighbors in f, then assign the color c to all vertices in $f \cup \{v'\}$.
- If v' has some neighbor in f and there exists $k \in K \setminus \{v\}$ with $kv' \notin E$, then assign the color c to the vertices k and v', and assign the color d to every vertex in f.
- If v' has some neighbor in f and v' is a neighbor of every vertex in $K \setminus v$, then $v' \notin K'$ implies the existence of $k' \in K'$ not adjacent to v' by the hypothesis (2). In this setting, assign the color c to the vertices k' and v', and assign the color d to every vertex in f.

Let $x^1 \in \{0,1\}^{|V||C|}$ represent the obtained coloring and define $w^1 = (x^1, e_f)$. We have $w^1 \in F$. Construct now $w^2 = (x^2, e_f)$ from w^1 by setting the color d' to the vertex v' and leaving the remaining color assignments unchanged. We again have $w^2 \in F$, implying $\lambda_{v'c} = \lambda_{v'd'}$. $\diamondsuit$
Claim 3: $\boldsymbol{\lambda_{v'd} = \lambda_{v'd'}\ \forall v' \in f \setminus \{w\}\ \forall d,\ d' \in C\backslash\{c\},\ d \neq d'}$**.** Consider any solution using $\chi(G)$ colors, such that no vertex is assigned the color d nor the color c, and such that v' is assigned the color d' (such a construction is possible by renaming the colors from any optimal coloring, since $\chi(G) + 1 < |C|$). Construct a new coloring from this solution by assigning the color c to the vertex w and to a vertex $k \in K$, which is possible by the hypothesis (1)). Call $w^1 = (x^1, \hat{0})$ the obtained solution, which belongs to F. Finally, construct the solution $w^2 = (x^2, \hat{0})$ from w^1 by assigning the color d to the vertex v', and note that $w^2 \in F$ again. The existence of w^1 and w^2 in F settles the claim. $\diamondsuit$

Claim 4: $\boldsymbol{\lambda_{v'c} = \lambda_{v'd'}}$ $\forall \boldsymbol{v' \in f \setminus \{w, v\}}$, $\forall \boldsymbol{d' \in C \setminus \{c\}}$. Again, consider any solution using $\chi(G)$ colors, and such that no vertex is assigned the color d' nor the color c. Modify this solution by assigning the color c to the vertices w and v', and by assigning the color c to a vertex $k \in K \setminus \{v\}$ such that $v'k \notin E$ (which exists by the hypothesis (3) applied to v'). Recall that hypothesis (1) also implies that w has no neighbors in K. Let $w^1 = (x^1, \hat{0})$ represent this solution, and note that $w^1 \in F$. Construct $w^2 = (x^2, \hat{0})$ from w^1 by assigning the color d' to the vertex v'. Again we have $w^2 \in F$, thus implying $\lambda_{v'c} = \lambda_{v'd'}$. $\Diamond$
Claim 5: $\boldsymbol{\lambda_{vc} = \lambda_{vd}}$ $\forall \boldsymbol{d \in C \setminus \{c\}}$. Consider a coloring with $\chi(G)$ colors, such that the colors c and d are not assigned to any vertex. Let $k' \in K'$, and modify the coloring so that the vertices v, w, and k' receive the color c (which is possible by the hypotheses (b) and (4)), and the remaining color assignments are left unchanged. Let $w^1 = (x^1, \hat{0})$ represent this solution, and construct $w^2 = (x^2, \hat{0})$ from w^1 by assigning the color d to the vertex v and keeping the remaining variables unchanged. Again, we have $w^1, w^2 \in F$, thus implying the claim. $\Diamond$
Claim 6: $\boldsymbol{\lambda_{ud} = \lambda_{ud'}}$ $\forall \boldsymbol{u \in K'}$, $\forall \boldsymbol{d, d' \in C \setminus \{c\}}$. Consider a coloring with $\chi(G)$ colors, such that the colors c and d are not assigned to any vertex, and such that the vertex u receives the color d' (such a solution exists since $\chi(G) + 1 < |C|$). Construct the solution $w^1 = (x^1, e_f)$ from this coloring by assigning the color c to every vertex in f and leaving the remaining assignments unchanged. Construct $w^2 = (x^2, e_f)$ from w^1 by assigning the color d to the vertex u and leaving the remaining variables unchanged. Again, we have $w^1, w^2 \in F$, thus implying $\lambda_{ud} = \lambda_{ud'}$. $\Diamond$
Claim 7: $\boldsymbol{\lambda_{ud} = \lambda_{ud'}}$ $\forall \boldsymbol{u \in K \setminus \{v\}}$, $\forall \boldsymbol{d, d' \in C \setminus \{c\}}$. We omit the proof of this claim, since it is very similar to the proof of Claim 6. $\Diamond$
Claim 8: $\boldsymbol{\mu_f + \lambda_{wd} = \lambda_{wc}}$ $\forall \boldsymbol{d \in C \setminus \{c\}}$. Consider a coloring with $\chi(G)$ colors, such that the colors c and d are not assigned to any vertex. Modify this coloring by setting the color d to every vertex in f, and by assigning the color c to a vertex $k \in K \setminus \{v\}$. Denote this solution by $w^1 = (x^1, e_f)$, and note that $w^1 \in F$. Construct $w^2 = (x^2, \hat{0})$ from w^1 by assigning the color c to the vertex w and keeping the remaining variables unchanged (this is possible by the hypothesis (1)). We again have $w^2 \in F$, hence the claim follows. $\Diamond$
Claim 9: $\boldsymbol{\lambda_{k'c} + \lambda_{kd} = \lambda_{kc} + \lambda_{k'd}}$ $\forall \boldsymbol{k' \in K'}$, $\forall \boldsymbol{k \in K \setminus \{v\}}$, $\forall \boldsymbol{d \in C \setminus \{c\}}$. Let $d', d'' \in C \setminus \{c, d\}$, $d' \neq d''$ (note that $|K| \geq 2$ and $|C| > \chi(G)+1$ implies $|C| \geq 4$, so such a choice of colors is possible). Let $x \in \{0,1\}^{|V||C|}$ represent a coloring with $\chi(G)$ colors, such that the colors c and d are not assigned to any vertex, and such that $x_{kd''} = x_{k'd'} = 1$. We may assume

that k and k' are assigned distinct colors since $kk' \in E$ by the hypothesis (a). Construct x^1 from x by assigning the color d to every vertex in f, and the color c to k', and consider the solution $w^1 = (x^1, e_f) \in F$. Finally, construct $w^2 = (x^2, \hat{0})$ from w^1 by setting $x^2_{wc} = x^2_{k'd'} = x^2_{kc} = 1$. We again have $w^2 \in F$, hence $\lambda_{k'c} + \mu_f + \lambda_{wd} + \lambda_{kd''} = \lambda_{wc} + \lambda_{kc} + \lambda_{k'd'}$. By Claim 8 we have $\lambda_{k'c} + \lambda_{kd''} = \lambda_{kc} + \lambda_{k'd'}$, and Claims 6 and 7 imply $\lambda_{k'd'} = \lambda_{k'd}$ and $\lambda_{kd''} = \lambda_{kd}$, thus $\lambda_{k'c} + \lambda_{kd} = \lambda_{kc} + \lambda_{k'd}$, settling the claim. $\diamondsuit$

Claim 10: $\boldsymbol{\mu_f + \lambda_{kd} = \lambda_{kc}\ \forall k \in K \setminus \{v\},\ \forall d \in C\setminus\{c\}}$**.** Consider a coloring with $\chi(G)$ colors such that the colors c and d are not assigned to any vertex. Modify this coloring by assigning the color c to every vertex from f, and by assigning the color d to the vertex k. Let $w^1 = (x^1, e_f)$ represent this feasible solution. Construct now $w^2 = (x^2, \hat{0})$ from w^1 by assigning the color d to every vertex in $f \setminus \{w\}$, and the color c to the vertex k (which is possible by the hypothesis (1)). We again have $w^2 \in F$, hence

$$\mu_f + \sum_{u \in f \setminus \{w\}} \lambda_{uc} + \lambda_{kd} = \sum_{u \in f \setminus \{w\}} \lambda_{ud} + \lambda_{kc}. \tag{10}$$

Claim 4 asserts that $\lambda_{uc} = \lambda_{ud}$ for every $u \in f \setminus \{v, w\}$, hence (10) reads $\mu_f + \lambda_{vc} + \lambda_{kd} = \lambda_{vd} + \lambda_{kc}$. Claim 5 implies $\lambda_{vc} = \lambda_{vd}$, so we get $\mu_f + \lambda_{kd} = \lambda_{kc}$, thus settling the claim. $\diamondsuit$

Claim 11: $\boldsymbol{\mu_f + \lambda_{k'd} = \lambda_{k'c}\ \forall k' \in K',\ \forall d \in C\setminus\{c\}}$**.** We get this result by combining Claims 9 and 10. $\diamondsuit$

These claims imply that (λ, μ) is a linear combination of the model constraints (2) and the coefficient vector of (8), hence F is a facet of $\mathcal{P}_{HMIC}(G, H, C)$. $\square$

We conjecture the separation problem associated with the union of cliques inequalities to be NP-complete, due to the maximum-clique structure present in the formulation of this problem.

3.3. Course-clique inequalities

Given a graph $G = (V, E)$ and a hypergraph $H = (V, \mathcal{F})$, we define $G_{\mathcal{F}} = (\mathcal{F}, E_{\mathcal{F}})$ to be the graph with one vertex for each hyperedge in H, and with an edge between any two vertices $f \in \mathcal{F}$ and $f' \in \mathcal{F}$ if there exist $i \in f$ and $i' \in f'$ with $ii' \in E$, representing that the corresponding hyperedges in H are adjacent in $G_{\mathcal{F}}$ if they contain vertices connected by an edge in G. In other words, the graph $G_{\mathcal{F}}$ represents "adjacencies in G" between hyperedges in H, indicating which pairs of hyperedges from H contain at least one vertex each that are neighbors in G. A set $Q \subseteq \mathcal{F}$ is called a *course clique* if Q is a clique in $G_{\mathcal{F}}$. We say that a hyperedge $f \in \mathcal{F}$

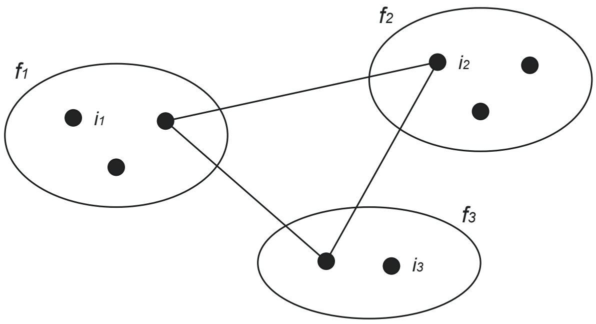


Figure 4: Example support of the course-clique inequality (11) for $k = 3$. Solid lines represent edges in G, and ellipses represent hyperedges in H.

is assigned the color $c \in C$ by a certain coloring if all the vertices from f receive the color c in the coloring.

Definition 5. *Let* $Q = \{f_1, \ldots, f_k\}$ *be a course clique. For* $j = 1, \ldots, k$*, fix a vertex* $i_j \in f_j$ *(see Figure 4). Finally, let* $D \subset C$ *with* $|D| = |C| - (k-1)$*. We define*

$$\sum_{j=1}^{k} z_{f_j} \leq k - 1 + \sum_{d \in D} \sum_{j=1}^{k} x_{i_j d} \tag{11}$$

to be the course-clique inequality *associated with* Q*,* D*, and* $\{i_1, \ldots, i_k\}$*.*

Theorem 3. *The course-clique inequality (11) is valid for* $\mathcal{P}_{HMIC}(G, H, C)$*. Moreover, if*

1. *there exists a feasible solution* $(\bar{x}, \bar{z})$ *using no more than* $|C| - 2$ *colors and such that* $\bar{z}_{f_1} = \ldots = \bar{z}_{f_k} = 1$*,*
2. *for every vertex* $v \in f_j$*,* $j = 1, \ldots, k$*, there exists* $\ell \in \{1, \ldots, j-1, j+1, \ldots, k\}$ *such that* v *has no neighbors in* f_l*, and*
3. *for every vertex* $i \notin f_1 \cup \ldots \cup f_k$*, there exists* $j \in \{1, \ldots, k\}$ *such that* i *has no neighbors in* f_j*,*

then (11) induces a facet of $\mathcal{P}_{HMIC}(G, H, C)$*.*

Proof. We first show that (11) is a valid inequality. To this end, suppose there exists an integer solution $(x, z) \in \mathcal{P}_{HMIC}(G, H, C)$ violating (11).

Since $\sum_{j=1}^{k} z_{f_j} \leq k$, we must have $\sum_{j=1}^{k} z_{f_j} = k$ and $\sum_{d \in D} \sum_{j=1}^{k} x_{i_j d} = 0$. This implies that the vertices $i_1, \ldots, i_k$ receive colors from $C \setminus D$. The fact that $|C \setminus D| = k - 1$ implies that at least two of these vertices, say $i_{j_1} \in f_{j_1}$ and $i_{j_2} \in f_{j_2}$, get the same color. Furthermore, $z_{f_1} = \ldots = z_{f_k} = 1$ implies that all vertices in f_{j_1} (respectively f_{j_2}) are assigned the same colors. This contradicts the fact that Q is a course clique, namely there exists an edge with endpoints in f_{j_1} and f_{j_2}, which are assigned the same color. Therefore, (11) is a valid inequality for $\mathcal{P}_{HMIC}(G, H, C)$.

Now for facetness. Let F be the face of $\mathcal{P}_{HMIC}(G, H, C)$ induced by (11). We first observe that the facetness hypothesis 1 implies the existence of a particular solution in F, that we will identify in the following claim.

Claim 0: There exists a feasible solution $\hat{w} = (\hat{x}, \hat{z}) \in F$ with $\hat{z}_{f_1} = \ldots = \hat{z}_{f_k} = 1$ and such that at least two colors $d_1, d_2 \in D$ are not assigned to any vertex. Let $C \setminus D = \{c_1, \ldots, c_{k-1}\}$. Consider the solution $(\bar{x}, \bar{z})$ proposed by the facetness hypothesis 1. Rename the colors in such a way that all vertices in f_j are assigned the color c_j, for $j = 1, \ldots, k-1$ (this is possible since all vertices in f_j are assigned the same color in $(\bar{x}, \bar{z})$, for $j = 1, \ldots, k$). Call $(\hat{x}, \hat{z})$ the obtained solution, with $\hat{z} := \bar{z}$. Since all the colors from $C \setminus D$ are assigned to some vertex and there are two unused colors, we can assume that the colors $d_1, d_2 \in D$ are not assigned to any vertex in $\hat{x}$. Since $\hat{z}_k = \bar{z}_k = 1$, all vertices in f_k are assigned the same color, which furthermore belongs to D. This implies that $(\hat{x}, \hat{z})$ satisfies (11) with equality. $\Diamond$

Suppose $\lambda^\intercal x + \mu^\intercal z = \lambda_0$ for every $(x, z) \in F$. We shall show that (λ, μ) is a linear combination of the model constraints (2) and the coefficient vector of (11), thus showing that F is a facet of of $\mathcal{P}_{HMIC}(G, H, C)$. Throughout the following claims we shall call $\hat{w} = (\hat{x}, \hat{z})$ the solution in F identified by Claim 0, and we shall call d_1 and d_2 the two colors in D not used in this solution.

Claim 1: $\mu_{f'} = 0$ $\forall f' \in \mathcal{F}$, $f' \neq f_1, \ldots, f_k$. Construct the solution $w^1 = (x^1, \sum_{j=1}^{k} e_{f_j})$ from $\hat{w}$ by assigning the color d_1 to every vertex in f'. Construct also the solution $w^2 = (x_1, e_{f'} + \sum_{j=1}^{k} e_{f_j})$. Since $w^1, w^2 \in F$, we get $\mu_{f'} = 0$. $\Diamond$

Claim 2: $\lambda_{i_j c} = \lambda_{i_j d} + \mu_{f_j}$ $\forall j \in \{1, \ldots, k\}$, $\forall c \in C \setminus D$, $\forall d \in D$. The solution $\hat{w}$ assigns all colors in $C \setminus D$, together with a color $d' \in D$, to the vertices in $f_1 \cup \ldots \cup f_k$. Construct a new coloring $\hat{x}^1$ from $\hat{x}$ by renaming the colors in such a way that all vertices from f_j are assigned the color d, while the remaining hyperedges from Q, namely $f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_k$, are assigned colors in $C \setminus D$. At least one color from D, say d_1', is not used in this solution. Let $i_p \in \{i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_k\}$ such that $i_p i_j \notin E$ (such a vertex exists by the facetness hypothesis (2)). Rename the colors in such a way that i_p is assigned the color c, and in such a way that the vertices outside $f_1 \cup \dots \cup f_k$ with color c in $\hat{x}^1$ are assigned the color d_1'. Let $x^1 \in \{0,1\}^{|V||C|}$ represent this coloring, and define the solution $w^1 = (x^1, \sum_{m=1}^{k} e_{f_m})$, which belongs to F. Finally, construct a new solution $w^2 = (x^2, \sum_{m=1, m\neq j}^{k} e_{f_m})$ from w^1 by setting $x^2_{i_j c} = 1$ and leaving the remaining color assignments unchanged. We again have $w^2 \in F$, since $k - 1 = \sum_{m=1}^{k} z_{f_m}$. The existence of w^1 and w^2 in F implies $\lambda_{i_j c} = \lambda_{i_j d} + \mu_{f_j}$. $\Diamond$

Claim 3: $\boldsymbol{\lambda_{id} = \lambda_{ic}}$ $\forall \boldsymbol{i \in V}$, $\boldsymbol{i \notin f_1 \cup \dots \cup f_k}$, $\forall \boldsymbol{d \in D}$, $\forall \boldsymbol{c \in C \setminus D}$. Consider the solution $\hat{w}$ and rename the colors in such a way that the colors d and d_1 are not assigned to any vertex. Call $\tilde{w}$ this new solution. Since $i \notin f_1 \cup \dots \cup f_k$, the hypothesis (3) ensures that there exists $j \in \{1, \dots, k\}$ such that i has no neighbors in f_j. Since $\hat{w}$ satisfies (11) with equality, there is a single hyperedge in $\{f_1, \dots, f_k\}$ all of whose vertices are assigned a color from D in $\tilde{w}$, whereas the vertices in the remaining hyperedges from Q will be assigned colors in $C \setminus D$. Rename again the colors in such a way that all vertices in f_j are assigned the color c, and let $x^1 \in \{0,1\}^{|V||C|}$ represent the resulting coloring. Define $w^1 = (x^1, \sum_{m=1}^{k} e_{f_m})$, and note that $w^1 \in F$ since a single hyperedge from Q is assigned a color from D. Finally, construct a new solution $w^2 = (x^2, \sum_{m=1}^{k} e_{f_m})$ from w^1 by assigning the color d to the vertex i and keeping the remaining color assignments unchanged. Since w^2 also belongs to F, the claim follows. $\Diamond$

Claim 4: $\boldsymbol{\lambda_{ic} = \lambda_{id'}}$ $\forall \boldsymbol{c \in C \setminus D}$, $\forall \boldsymbol{d' \in D}$, $\forall \boldsymbol{i \in f_j, i \neq i_j}$, $\boldsymbol{j = 1, \dots, k}$. Let $d, d'' \in D \setminus \{d'\}$, $d \neq d''$ (note that $|D| \geq 3$ since one color from D is used in $\hat{w}$ and at least two other colors from D are left unused in this solution). Let $f_{n'}$ and f_n be hyperedges such that i_j has no neighbors in $f_{n'}$, and i has no neighbors in f_n. We shall construct a solution from $\hat{w}$ with the following procedure.

1. Consider the solution $\hat{w}$ and rename the colors in such a way that d' and d'' are not assigned to any vertex.
2. Rename the colors in such a way that f_n is assigned the color c. By the construction of $\hat{w}$, there exists some hyperedge in $\{f_1, \dots, f_k\}$ all of whose vertices get the color c, hence in this step it suffices to rename the colors assigned to two hyperedges from Q.
3. Exactly one hyperedge from Q is assigned a color from D. Rename the colors in such a way that f_j is this hyperedge, and that the assigned color from D is precisely d. In other words, rename the colors so that all the vertices from f_j are assigned the color d.

4. Let $c_1 \in C \setminus D$ be the color assigned to $f_{n'}$ (recall that exactly one hyperedge from Q is assigned a color from D), and assign this color to the vertex i_j. In order not to lose feasibility, assign the color d'' to all vertices outside Q assigned the color c_1.
5. Assign the color d' to all vertices outside Q receiving the color c.

Let $x^1 \in \{0,1\}^{|V||C|}$ represent the obtained coloring, and construct the solution $w^1 = (x^1, \sum_{m=1, m\neq j}^{k} e_{f_m})$. We have $w^1 \in F$, since $\sum_{t=1}^{k} z_{f_t} = k-1$ and i_j is assigned a color in $C \setminus D$. Finally, construct a new solution $w^2 = (x^2, \sum_{m=1, m\neq j}^{k} e_{f_m})$ from w^1 by assigning the color c to the vertex i and keeping the remaining assignments unchanged. Again, $w^2 \in F$, and the claim follows. $\Diamond$

Claim 5: $\boldsymbol{\lambda_{i_j d} + \lambda_{i_m c} = \lambda_{i_m d} + \lambda_{i_j c} \ \forall j, m \in \{1, \ldots, k\}, d \in D, c \in C \setminus D}$**.** Consider the solution $\hat{w}$ and rename the colors in such a way that all vertices in f_j are assigned the color d and all vertices in f_m are assigned the color c. Such a configuration is possible since at least one hyperedge from $\{f_1, \ldots, f_k\}$ has all its vertices assigned some color from D. Also, let $d_1 \in D$ and $d_2 \in D$ be the colors not assigned to any vertex. Now, assign the color d_1 to all vertices outside Q previously assigned the color c, and assign the color d_2 to all vertices outside Q previously assigned the color d. Represent the resulting solution by $w^1 = (x^1, \sum_{m=1}^{k} e_{f_m})$. Finally, exchange the colors assigned to f_m and f_j, so that all the vertices in f_j are assigned the color c and all the vertices in f_m are assigned the color d. Represent the resulting solution by $w^2 = (x^2, \sum_{m=1}^{k} e_{f_m})$. We have $w^1, w^2 \in F$, implying

$$\sum_{i \in f_m} \lambda_{id} + \sum_{i \in f_j} \lambda_{ic} = \sum_{i \in f_m} \lambda_{ic} + \sum_{i \in f_j} \lambda_{id}. \tag{12}$$

Claim 4 implies $\sum_{i \in f_m \setminus \{i_m\}} \lambda_{id} = \sum_{i \in f_m \setminus \{i_m\}} \lambda_{ic}$, and a similar assertion holds for f_j and i_j. Therefore, (12) simplifies to $\lambda_{i_m d} + \lambda_{i_j c} = \lambda_{i_m c} + \lambda_{i_j d}$, thus settling the claim. $\Diamond$

Claim 6: $\boldsymbol{\mu_{f_i} = \mu_{f_j} \ \forall i, j \in \{1, \ldots, k\}}$**.** This claim follows directly from Claims 2 and 5. $\Diamond$

These claims imply that (λ, μ) is a linear combination of the model constraints (2) and the coefficient vector of (11), hence F is a facet of $\mathcal{P}_{HMIC}(G, H, C)$. $\Box$

We conjecture the separation problem associated with the course-clique inequalities to be NP-complete, again due to the maximum-clique structure in the graph $G_{\mathcal{F}}$ present in the formulation of this problem.

4. Computational experiments

We report in this section our computational experiments in order to evaluate the contribution of the families of valid inequalities presented in Section 3 to the practical resolution of real instances of the maximum-impact coloring problem on hypergraphs. We first describe the computational procedures for identifying violated valid inequalities within a cutting plane environment. All the separation procedures take as input the instance data (i.e., the graph G, the hypergraph H, and the color set C) and the fractional solution $w^* = (x^*, z^*) \in \mathbb{R}^{|V||C|+|\mathcal{F}|}$ to be separated. In this context, we say that a hyperedge $f \in \mathcal{F}$ is *fractional* if $\varepsilon < x^*_{ic} < 1 - \varepsilon$ for some $i \in f$ and some $c \in C$, or if $\varepsilon < z^*_f < 1 - \varepsilon$. We take $\varepsilon = 0.01$ in our implementation. All procedures also employ a threshold $\tau \in [0, 1]$ for considering a variable in the procedure, and a minimum cut depth $\rho \in \mathbb{R}$ in order to consider a valid inequality to be violated. We set $\tau = 0.25$ and $\rho = 0$, unless otherwise stated.

Partition inequalities with $n = 2$ (P). This procedure searches for partition inequalities (7) with $n = 2$, i.e., involving two vertices from the hyperedge f and a 2-partition of the color set C. Algorithm 1 details the procedure, which considers all pairs of vertices within each fractional hyperedge. If $\tau = 0$ then this procedure is exact, i.e., if there is a violated partition inequality with $n = 2$, then this procedure finds such an inequality. This assertion stems from the fact that the "for" loop in lines 5–9 searches for the partition $\{D, C\backslash D\}$ minimizing the right-hand-side of (7). The overall procedure runs in $O(|\mathcal{F}||C|\omega^2)$ time, where $\omega = \max_{f\in\mathcal{F}} |f|$.

Partition inequalities with $n = |f|$ (GP). This procedure searches for partition inequalities (7) with n equal to the number of vertices in the hyperedge f associated with the inequality, i.e., $\mathcal{F} = \{i_1, \ldots, i_n\}$. Algorithm 2 details the procedure which, for each hyperedge $f \in \mathcal{F}$, constructs a partition $\{D_1, \ldots, D_{|f|}\}$ of C minimizing the right-hand-side of (7). Again, for $\tau = 0$ such a construction guarantees to find violated partition inequalities with $n = |f|$, if such inequalities exist. It may be the case that $D_k = \emptyset$ for some $k \in \{1, \ldots, n\}$ upon termination of the procedure. If this is the case and a violated inequality is found, then the generated cut has $n < |f|$. The overall procedure runs in $O(|\mathcal{F}||C|\omega)$ time.

Union of cliques inequalities (UC). We greedily separate the union of cliques inequalities (8), and we restrict ourselves to considering such inequalities for $K' = \emptyset$ in order to simplify the separation procedure. For each fractional hyperedge $f \in \mathcal{F}$ with $z^*_f \geq \tau$, for each color $c \in C$, and for each $v, w \in f$, $v \neq w$, we greedily construct a clique $K \subseteq (V\backslash f) \cup \{v\}$

Algorithm 1 Separation for the partition inequalities with $n = 2$.

for $f \in \mathcal{F}$ **do**
    if $z_f^* \geq \tau$ and f is fractional **then**
        for $i, j \in f$, $i \neq j$ **do**
            $D \leftarrow \emptyset$
            for $c \in C$ **do**
                if $x_{ic}^* > x_{jc}^*$ **then**
                    $D \leftarrow D \cup \{c\}$
                end if
            end for
            if w^* violates (7) associated with i, j, and D **then**
                Add the inequality to the cut pool
            end if
        end for
    end if
end for

Algorithm 2 Separation for the partition inequalities with $n = |f|$.

for $f \in \mathcal{F}$ **do**
    if $z_f^* \geq \tau$ and f is fractional **then**
        $D_1, \ldots, D_{|f|} \leftarrow \emptyset$
        for $c \in C$ **do**
            $k \leftarrow \operatorname{argmin}\{x_{ic}^* : i \in f\}$
            $D_k \leftarrow D_k \cup \{c\}$
        end for
        if w^* violates (7) associated with i, j, and D **then**
            Add the inequality to the cut pool
        end if
    end if
end for

by considering the vertices in $V \backslash f$ in decreasing value of x^*_{ic}. This heuristic procedure runs in $O(|\mathcal{F}||C||V|^2\omega^2)$ time, where $\omega = \max_{f\in\mathcal{F}} |f|$.
Course-clique inequalities with $|Q| = 2$ (2C). In order to avoid the computational burden of dealing with large course cliques, we separate with this heuristic procedure the couse-cliques having just two constituent cliques, i.e., $|Q| = 2$, hence D must be a singleton in order to satisfy the validity hypotheses. To this end, we first precompute all pairs of adjacent vertices in $G_{\mathcal{F}}$, namely the set $E_{\mathcal{F}}$. For each $fg \in E_{\mathcal{F}}$ and each color $c \in C$, we take $j = \text{argmax}\{x^*_{ic} : i \in f\}$ and $k = \text{argmax}\{x^*_{ic} : i \in g\}$. If the course-clique inequality (11) associated with $Q = \{f, g\}$, $D = \{c\}$, and $\{i_1 := j, i_2 := k\}$ is violated, then we add such an inequality to the cut pool. Once $E_{\mathcal{F}}$ is precomputed, this heuristic procedure runs in $O(|E_{\mathcal{F}}||C|\omega)$ time, where $\omega = \max_{f\in\mathcal{F}} |f|$.
Course-clique inequalities with $|Q| = 3$ (3C). This procedure is similar to the previous one, now considering triangles instead of edges in $G_{\mathcal{F}}$. To this end, we first precompute the set $\mathcal{T} = \{\{f_1, f_2, f_3\} : f_1, f_2, f_3 \in \mathcal{F}, f_1f_2 \in E_{\mathcal{F}}, f_2f_3 \in E_{\mathcal{F}}, f_1f_3 \in E_{\mathcal{F}}\}$. For each $\{f_1, f_2, f_3\} \in \mathcal{T}$ and each $c, d \in C$, $c_1 \neq c_2$, we find $i_t = \text{argmax}\{x^*_{jc} + x^*_{jd} : j \in f_t\}$, for $t = 1, 2, 3$. If the inequality (11) associated with $Q = \{f_1, f_2, f_3\}$, $D = \{c, d\}$, and $\{i_1, i_2, i_3\}$ is violated, then we add such an inequality to the cut pool. Once $\mathcal{T}$ is precomputed, this heuristic procedure runs in $O(|\mathcal{T}||C|^2\omega)$ time, where $\omega = \max_{f\in\mathcal{F}} |f|$.

We have implemented a branch and cut procedure within the framework provided by Cplex 12.4. The controller code is implemented in Java, interfacing with Cplex through the Concert API [4]. The described separation procedures were implemented using standard Java libraries. Table 1 presents a first assessment of the contribution of these cuts within the procedure implemented by Cplex. The first five columns provide data on each instance, whereas the following two groups of columns report the performance of Cplex with out-of-the-box parameters without and with the addition of cuts coming from the separation procedures, respectively. The column "Time/Gap" reports the running time to optimality and, if optimality is not achieved within 10 minutes, the optimality gap between brackets. The column "Nodes" reports the number of nodes in the enumeration tree upon termination of the branch and cut procedure.

In these experiments we use the instances described in [1], interpreting the data for the maximum-impact coloring problem on hypergraphs (i.e., we create an hyperedge for each set of lectures belonging to the same course). Since the original instances are quite easy to solve for Cplex, we have also constructed some instances from them by adding a small number

Instance					Cplex		Cplex + cuts	
Name	$\|V\|$	$\|E\|$	$\|\mathcal{F}\|$	$\|C\|$	Time/Gap	Nodes	Time/Gap	Nodes
10-1-p1a	235	1894	108	21	1.44	0	1.6	0
10-1-p1c	235	1894	113	21	40.9	198	45.17	77
10-1-p2a	267	2593	144	22	3.47	0	3.74	0
10-1-p2d	267	2593	150	22	[3.96%]	169	335.38	22
10-2-p1a	256	1884	106	18	34.21	2857	22.04	143
10-2-p1c	256	1884	111	18	[2.40%]	21800	194.95	742
10-2-p1d	256	1884	112	18	[2.77%]	9088	[3.84%]	613
10-2-p2d	279	2914	122	26	[4.07%]	1663	456.47	616
11-1-p1a	265	2092	113	16	1.19	0	1.32	0
11-1-p1c	265	2092	118	16	[1.40%]	3339	235.53	542
11-1-p2c	255	2295	127	20	210.75	187	78.62	5
11-1-p2d	255	2295	128	20	422.71	875	454.51	47
12-1-p1c	182	1381	91	18	230.66	4723	150.88	198
12-1-p2a	235	2220	119	23	71.25	423	32.76	88
12-2-p1a	253	1974	129	20	1.44	0	1.58	0
12-2-p1d	253	1974	135	20	101.92	129	15.98	3
12-2-p2b	254	2368	140	22	173.06	155	91.12	18
12-2-p2c	254	2368	141	22	342.16	315	299.31	51
12-2-p2d	254	2368	142	22	[2.64%]	985	183.21	22
14-2-p1a	172	1201	104	20	71.76	1060	78.15	316
14-2-p1b	172	1201	108	20	[3.37%]	9252	468.53	260
14-2-p1c	172	1201	109	20	[5.51%]	1830	223.23	52
14-2-p1d	172	1201	110	20	[3.92%]	8890	31.21	3
14-2-p2c	238	2294	167	20	[7.58%]	144	[8.32%]	16

Table 1: Performance of Cplex with and without the dynamical addition of cuts from the families of valid inequalities identified in this work.

of randomly-generated hyperedges which, in our experience, greatly complicated running times to optimality. This procedure may generate an instance violating Assumption 2 but, since this assumption only applies to facetness and does not affect validity, we can safely use the families of valid inequalities identified in this work. It can be seen that the addition of the inequalities consistently reduces the number of nodes in the enumeration tree, although not always this corresponds to a reduction of the total running time to optimality, due to the overhead introduced by the separation procedures and larger linear relaxations. However, for several instances the addition of cuts allows either to reduce running times with respect to Cplex, or to prove optimality within the time limit.

Table 2 reports several statistics of the separation procedures. All separation procedures generate cuts, the course-clique inequalities with $|Q| = 3$ being the most prolific in our experiments. The effectivenes at finding violated inequalities seems to be acceptable, with the exception of the separa-

Concept	P	GP	UC	2C	3C
Average number of cuts per instance	40.17	19.79	9.38	3.29	2478.42
Maximum number of cuts per instance	194	85	25	16	12052
Average number of cuts per execution	1.35	0.47	0.55	0.13	108.32
Average number of cuts per node	1.13	0.41	0.45	0.12	100.03
Instances with no generated cuts	4	4	5	9	4
Separation time vs total time	$< 0.01\%$	$< 0.01\%$	0.29%	0.02%	6.06%
Variation in time when removed	-2.71%	-0.99%	0.38%	-14.35%	+6.91%
Variation in nodes when removed	+4.27%	-2.65%	-4.18%	-15.07%	+138.82%

Table 2: Statistics of the separation procedures.

tion procedure for the course-clique inequalities with $|Q| = 2$, which finds a cut roughly once every 8 executions on average (by "execution" we mean a single run of the separation procedure which, depending on the parameter setting, may take place more than once at each node in the enumeration tree). The separation time, expressed as a percentage of the total running time, seems negligible for all procedures with the exception of the course-clique inequalities with $|Q| = 3$. This observation opens up a line for further research, since the proposed separation procedure for this last family seems to find many cuts while employing a sizeable portion of the total running time, so it would be interesting to devise ways to limit its search space while maintaining its effectiveness.

The last two lines in this table report the variation in the total time to optimality and in the total number of nodes when each of the separation procedures is deactivated. Here, a negative figure implies that the removal of the corresponding separation procedure is *beneficial* to the overal algorithm, suggesting that separating the corresponding familiy does not pay off. The most dramatic figures in these rows correspond to the course-clique inequalities with $|Q| = 2$ and $|Q| = 3$, the former being quite detrimental for the overall performance, while the latter being crucial to the success of the complete procedure.

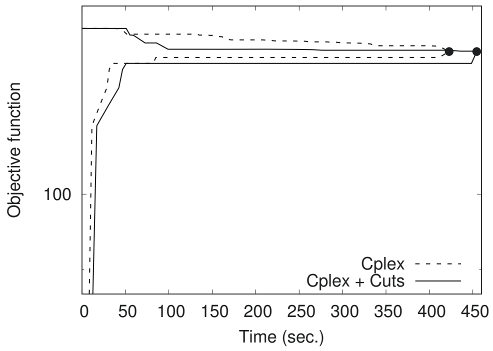
It is interesting to compare the convergence of the procedure augmented with the cuts proposed in this work with the convergence of Cplex. To this end, Figure 5 reports the lower (i.e., primal) and upper (i.e., dual) bounds of Cplex with and without the addition of the cuts identified in this work as a function of the running time for three representative instances. Figure 5(a) presents an instance for which the addition of cuts is slightly detrimental to the overall performance, whereas Figures 5(b) and (c) report instances in which the addition of cuts improves the performance. As expected, these figures show that the dual bound is greatly improved by the addition of the cuts identified in this work, and this helps to close the gap between the primal and the dual bounds faster. This behavior is consistently observed in most of the instances considered in this work. It is interesting to note that the addition of cuts also improves the dual bound for the instance 11-1-p2d (Figure 5(a)) but, in this case, the branch and cut procedure fails to find the optimal solution fast enough, and so the overall running time is larger for this instance.

As mentioned before, in the separation procedures we discard a violated inequality $\pi x + \mu z \leq \pi_0$ if $\pi x^* + \mu z^* \leq \pi_0 + \rho$, i.e., we impose a minimum cut depth ρ in order to discard slightly violated inequalities. Figure 6 presents the behavior of the procedure for a set of selected and optimally-solvable instances as a function of this parameter. In order to compare among different instances, the results are normalized by the measurements obtained for $\rho = 0$. As expected, the total number of cuts (Figure 6(c)) increases as the parameter ρ decreases (note that $\rho < 0$ corresponds to accepting non-violated valid inequalities into the cut pool) and the number of nodes in the enumeration tree (Figure 6(b)) tends to decrease. However, adding too many non-violated inequalities is detrimental for the overall running time (Figure 6(a)), and $\rho \in [0.2, 0.8]$ would seem to be the best choice for this parameter.

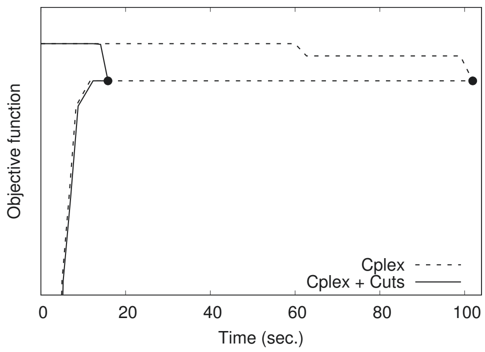
As observed in the previous experiments, a larger number of cuts does not imply a better running time, due to the overhead imposed by the resolution of larger linear relaxations. This balance can be tuned with the parameter called *skip factor* (SF), which determines every which number of nodes in the enumeration tree the separation procedures should be run. A value $SF = 1$ implies that the separation procedures are applied in all the nodes, whereas, e.g., a value $SF = 5$ implies that one every five nodes are subject to cutting rounds. Again and as expected, the total number of cuts (Figure 7(c)) clearly decreases and the number of nodes slightly increases (Figure 7(b)) as the skip factor increases. The effect on the total running time (Figure 7(a)) is not as marked, with a slight tendency to achieve better running times for $SF \geq 5$.

5. Concluding remarks

In this work we have started a polyhedral study of the maximum-impact coloring on hypergraphs, motivated by a similar previous work on graphs. Real instances coming from the University of Buenos Aires seem quite



(a)



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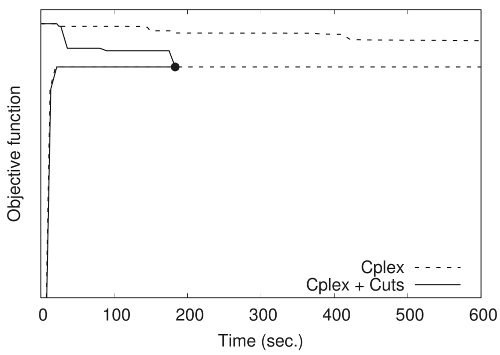
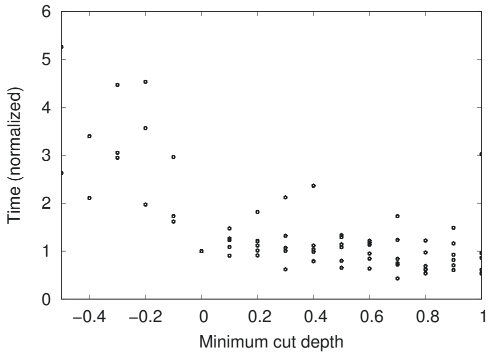
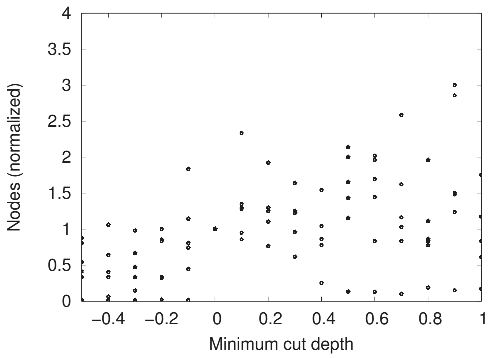


(c)

Figure 5: Lower and upper bounds obtained by Cplex (dashed lines) and obtained by Cplex with the addition of the cuts identified in this work (solid lines) as a function of the running time for the instances (a) 11-1-p2d, (b) 12-2-p1d, and (c) 12-2-p2d (vertical axis in logarithmic scale). Solid circles indicate the termination of the procedure.



(a)



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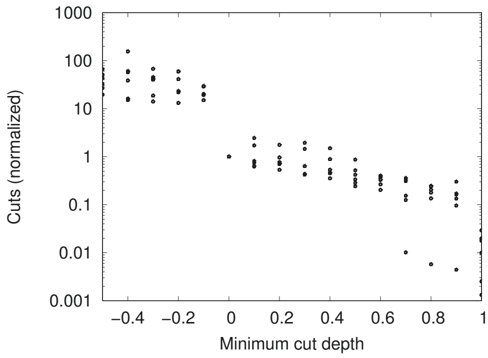
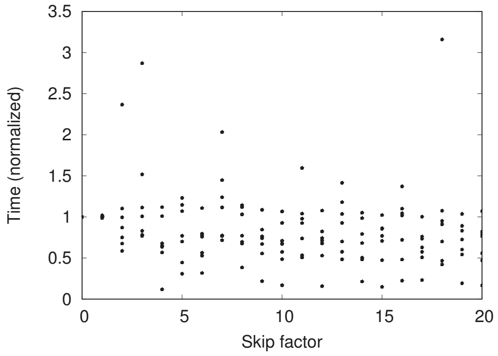
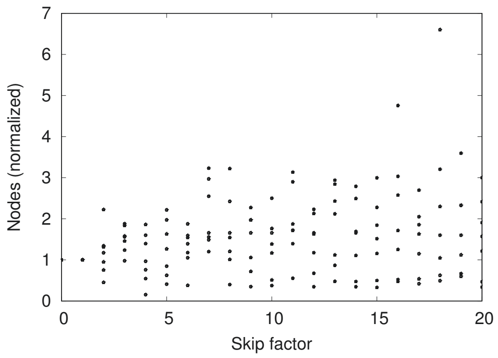


(c)

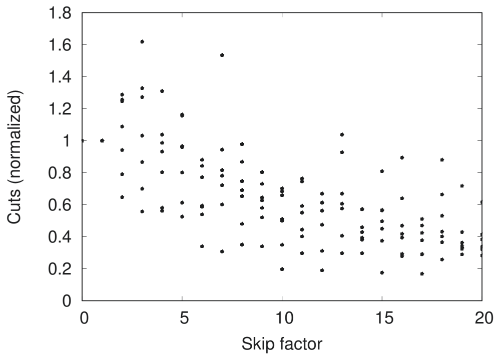
Figure 6: (a) Overall time to optimality, (b) number of nodes in the enumeration tree, and (c) number of generated cuts as a function of the minimum cut depth ρ for selected instances, and normalized with respect to the measurements for $\rho = 0$.



(a)



(b)



(c)

Figure 7: (a) Overall time to optimality, (b) number of nodes in the enumeration tree, and (c) number of generated cuts as a function of the skip factor (SF), and normalized with respect to the measurements for $SF = 1$.

straightforward to solve with optimality, but the computational difficulty seems to quickly escalate as soon as additional hyperedges are added. Our results show that it is possible to identify facet-inducing inequalities for an integer programming formulation, albeit with the addition of hypotheses needed to ensure faceteness. According to our computational experiments, the families of valid inequalities identified in this work may contribute to improve running times to optimality within a commercial solver.

Our experiments suggest that the course-clique inequalities over triangles in $G_{\mathcal{F}}$ may provide the most effective cuts in practice. Solving the separation problem for these inequalities is not straightforward and is the most expensive procedure in our implementation, so it would be interesting to search for more efficient ways of separating these inequalities. Also, the large number of generated inequalities from this family may not be beneficial to the overall procedure, and there may be room for improvement in practice by further tuning this procedure. It is interesting to note that these inequalities involve a "complicated" structure in the hypergraph H, in contrast to the other families being separated, which consider just a single hyperedge from H or a single edge in $G_{\mathcal{F}}$. This observation suggests that it might be a good idea to search for further valid inequalities involving nontrivial structures in $G_{\mathcal{F}}$, in order to make further progress at solving the maximum-impact coloring on hypergraphs in practice.

An interesting variation of the problem considered in this work consists in maximizing the number of courses such that all but one lecture are assigned to the same classroom. Such an assignment could be considered a reasonable solution, especially if the lecture assigned to a different classroom is of a different type than the other lectures. Unfortunately, the formulations considered in this work do not seem to accommodate this objective function in a straightforward way. Indeed, the definition of the z-variables in the formulation considered in this work heavily relies on just two lectures being assigned different colors in order to set the associated z-variable to a null value. It would be interesting to explore integer programming formulations for this problem, and whether similar techniques to the ones applied in this work could improve the solution times of a general solver.

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Data availability

Our implementation and the instances are available at the Github repository `https://github.com/jmarenco/CMIH`.

Statements and declarations

The authors declare that they have no conflict of interest.

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