Supermodular Utility Representations

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Abstract

Many problems in decision theory and game theory involve choice problems over lattices and invoke the assumption of supermodularity of utility functions. In the context of choice over finite lattices, it is well-known that existence of supermodular representations is equivalent to existence of quasisupermodular ones for monotone preferences. In particular, strictly monotone preferences admit a supermodular representation. This paper revisits the axiomatic foundations of supermodularity of utility functions representing preferences over finite lattices, and develops an axiomatic foundation in the context of choice over lotteries over outcomes in arbitrary lattices.

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1 Introduction

Supermodularity of payoff functions is a very convenient and increasingly popular assumption in many applications in economics. When the feasible set has a lattice structure and the objective function is supermodular, the tools of *monotone comparative statics* (Topkis (1978), Milgrom and Shannon (1994) and Topkis (1998)) provide a convenient means for making predictions about the parametric dependence of the solution set and of the value function. Another example is the class of *supermodular games* (Topkis (1979), Vives (1990) and Milgrom and Roberts (1990)) or of games with *(ordinal) strategic complementarities* (Milgrom and Shannon (1994)). These are games with strategy spaces endowed with a lattice structure and payoff functions that have certain supermodularity properties. Existence of pure strategy Nash equilibria in games in these classes is established by order-theoretic arguments. McAdams (2003) proves the existence of non-decreasing equilibria in games with incomplete information in which players' payoff functions have certain supermodularity properties.

However, axiomatic foundations for supermodularity remain incomplete. Echenique and Chambers (2009) identify a necessary and sufficient condition on preferences defined on finite lattices to admit a supermodular representation (Theorem 11). Their condition, however, is mainly technical and the economic intuition behind it is less than clear. They also present what can be read as another supermodular representation theorem (Theorem 1, a counterpart of Theorem 5 in Echenique and Chambers (2008)) with simpler conditions on preferences but also restricted to finite sets. Shirai (2010) analyzes the case of countably infinite lattices in $\mathbb{N}^n$ and focuses exclusively on submodularity (Theorem 2). His argument does not readily extend to arbitrary countable lattices and does not apply symmetrically to supermodularity, however.

Moreover, Echenique and Chambers (2009) also conclude that supermodularity is a vacuous assumption under finiteness and strict monotonicity, in the sense that a finite consumption data set from a finite consumption lattice can be rationalized if and only if it can be rationalized by a supermodular utility function (Corollary 5 and Proposition 7, respectively). In other words, supermodularity is 'a very weak assumption which is not testable with data on consumption expenditures'.[1]

[1]Echenique and Chambers (2009).

This paper revisits the axiomatic foundations of supermodularity of utility functions representing preferences over finite lattices, and develops an axiomatic foundation in the context of choice over lotteries over outcomes in arbitrary lattices. Section 2 revisits the work of Echenique and Chambers (2008), Echenique and Chambers (2009) and Shirai (2010) on supermodularity of representations of preferences over well-ordered sets. In addition, a 'supermodular counterpart' of the additive representation in Kreps (1979) is explored. Section 3 looks into the problem in the framework of choice over lotteries and develops axiomatic foundations for supermodularity of utility functions over outcomes in expected utility settings. Section 4 concludes. While the proofs of the main propositions are presented in the main body of the paper, some proofs are relegated to the appendix.

2 Supermodularity in Ordinal Frameworks

This section revisits the problem of axiomatizing supermodularity in an *ordinal utility* setting, that is, when preferences are defined over X. Such frameworks present a series of challenges. First, the order and the preference relation are defined on the same space, so supermodularity is a restriction on how these two coexisting relations interact. Consequently, axioms must be stated in terms of the same preference relation for which we want to produce a supermodular representation. Moreover, there is no generally accepted structure on the choice problem under which to attempt to construct supermodular representations, so even existence of representations is an issue. Finally, we can generally only aspire to produce 'a' supermodular representation, as supermodularity is typically not preserved under arbitrary strictly increasing transformations.

Lattices, monotonicity and supermodularity

Let X be a set and let $\gg \subseteq X \times X$ be a *partial order* on X, that is, a reflexive, transitive and antisymmetric binary relation on X, with asymmetric part $>$.[2] The pair $(X, \gg)$ is called

[2]The modifier 'partial' in partial order means that some pairs of elements in X may be incomparable under $\gg$. That is to say, there may be some $x_0, x_0' \in X$ such that neither $x_0' \gg x_0$ nor $x_0 \gg x_0'$. The

a *poset*. This poset is a *lattice* if for all $x, x' \in X$, both the supremum and the infimum under $\gg$, $\sup_{\gg}(\{x, x'\})$ and $\inf_{\gg}(\{x, x'\})$, exist and belong to X. Let $\vee, \wedge : X \times X \to X$ denote the 'join' and 'meet' operations on $(X, \gg)$, respectively: $x \vee x' := \sup_{\gg}(\{x, x'\})$ and $x \wedge x' := \inf_{\gg}(\{x, x'\})$. If $A \subseteq X$ has the property that the restrictions of $\vee, \wedge$ to $A \times A$, $\vee|_{A \times A}, \wedge|_{A \times A}$, map into A, then $(A, \gg_{A \times A})$ is called a *sublattice* of $(X, \gg)$, where $\gg_{A \times A}$ is the restriction of $\gg$ to $A \times A$, $\gg \cap A \times A$.[3]

A real-valued function $f : X \to \mathbb{R}$ is *non-decreasing* if, for any $x, x' \in X$, $x' \gg x \Rightarrow f(x') \geq f(x)$, while it is *strictly increasing* if $f(x') > f(x)$ whenever $x' > x$. If

$$f(x \vee x') + f(x \wedge x') \geq f(x) + f(x')$$

for any $x, x' \in X$, then $f : X \to \mathbb{R}$ is *supermodular*. . Thus, for a supermodular function, the total value of the meet and the join of any two points is at least as great as the total value of the two points individually. If the inequality is reversed, then f is *submodular*.

Following Li Calzi (1990) and Milgrom and Shannon (1994), $f : X \to \mathbb{R}$ satisfies the *downgrading property* or is *quasisupermodular* if, for any $x, x' \in X$:

$$f(x) \geq f(x \wedge x') \Rightarrow f(x \vee x') \geq f(x')$$

with the corresponding statement for strict inequality. Thus, for any $x, x' \in X$, if 'meeting' x with x' 'downgrades' x, then 'joining' x and x' 'upgrades' x' (according to f).[4] In other words, if the value of x under f is strictly higher than the value under f of $x \wedge x'$, then f cannot attain a higher value under x' than under $x \vee x'$. If the implications run in the other direction, then f is *quasisubmodular*.

A property $\mathcal{P}$ of a function $f : X \to \mathbb{R}$ is an *ordinal* property if $v \circ f : X \to \mathbb{R}$ also satisfies

statement $x' > x$ for $x, x' \in X$ means that $x' \gg x$ and $x' \neq x$.

[3]To simply notation, the restriction of a binary relation to a subset of the space will be denoted by the same symbol as the relation on the whole of the set in the sequel.

[4]Li Calzi (1990) presents this property in terms of the contrapositive statements: for any $x, x' \in X$:

$$f(x') \geq (>) f(x \vee x') \Rightarrow f(x \wedge x') \geq (>) f(x)$$

and uses the term 'downgrading'. The term 'quasisupermodularity' is from Milgrom and Shannon (1994).

$\mathcal{P}$ for any strictly increasing $v : \mathbb{R} \to \mathbb{R}$. Property $\mathcal{P}$ is instead a *cardinal* property of f if it is invariant to positive affine transformations of f; that is to say, if for any $a > 0$, $b \in \mathbb{R}$, the function $g = af + b$ also satisfies $\mathcal{P}$.[5] Of course, ordinal properties are also cardinal properties.[6]

Being non-decreasing or strictly increasing is itself an ordinal property. Supermodularity, on the other hand, is a cardinal property, while quasisupermodularity is an ordinal implication of supermodularity: any strictly increasing transformation of a supermodular function is quasisupermodular. In particular, any supermodular function is itself quasisupermodular. Since $x \gg x \wedge x'$ and $x \vee x' \gg x'$ for any $x, x' \in X$, any non-decreasing real-valued function on X automatically satisfies the first part of the property of quasisupermodularity. Moreover, any strictly increasing function is (automatically non-decreasing and) both quasisupermodular and quasisubmodular.[7]

Quasisupermodular preferences and supermodular representations

Let $\succsim \subseteq X \times X$ be a *complete preorder* on X, that is, a transitive binary relation on X such that all pair of elements from X are comparable under. Let the asymmetric and symmetric parts of $\succsim$ be denoted by $\succ$, $\sim$ respectively.[8] Let $L_{\succsim} : X \rightrightarrows X$ be the self-correspondence that assigns the $\succsim$-*lower contour* set to any $x \in X$: $L_{\succsim}(x) = \{\tilde{x} \in X : x \succsim \tilde{x}\}$. This correspondence is non-empty-valued and satisfies the following nesting property: $L_{\succsim}(x) \subseteq L_{\succsim}(x')$ whenever $x' \succsim x$.

A real-valued function $u : X \to \mathbb{R}$ *represents* $\succsim$ or is *a representation of* $\succsim$ if $x' \succsim x$ if and only if $u(x') \geq u(x)$ for any $x, x' \in X$. A real-valued function $u : X \to \mathbb{R}$ that represents $\succsim$ and has property $\mathcal{P}$ is a $\mathcal{P}$ *representation of* $\succsim$. If there exists a representation of $\succsim$, then $\succsim$ *admits a representation*. Similarly, if there exists a $\mathcal{P}$ representation of $\succsim$, then we say that $\succsim$ *admits a* $\mathcal{P}$ *representation*. It is clear that, for any strictly increasing function $g : \mathbb{R} \to \mathbb{R}$, the composite function $g \circ u$ represents $\succsim$ if u does. Therefore, the property of representing

[5]The function g is defined as $g(x) = af(x) + b$ for any $x \in X$.

[6]Clearly, the function $v : \mathbb{R} \to \mathbb{R}$ given by $v(s) = as + b$ is strictly increasing if $a > 0$.

[7]A proof of this well-known fact, corollary 5 in Echenique and Chambers (2009), is provided in the appendix as Lemma 1 for the sake of completeness.

[8]Thus, the statement $x' \succ x$ for $x, x' \in X$ means that $x' \succsim x$ but not $x \succsim x'$, while $x \sim x'$ means that both $x' \succsim x$ and $x \succsim x'$.

$\succsim$ is itself an ordinal property. On the other hand, if $\mathcal{P}$ is a cardinal property, then so is the property of being a $\mathcal{P}$ representation of $\succsim$.

Both binary relations $\gg$ and $\succsim$ are defined on X. As they are both (pre)orders, the triple $(X, \gg, \succsim)$ is a *doubly-ordered space*. Alternatively, we can think of preferences $\succsim$ being defined on the lattice $(X, \gg)$. We say that $\succsim$ is *monotone* if, for any $x, x' \in X$, $x' \gg x \Rightarrow x' \succsim x$. If $x' \succ x$ whenever $x' > x$, then $\succsim$ is *strictly monotone.*

Following Echenique and Chambers (2008) and Savvateev (2008), let $\succsim$ be *quasisupermodular* if:

$$x \succsim x \wedge x' \Rightarrow x \vee x' \succsim x',$$

with a corresponding statement for strict preference $\succ$. Under quasisupermodular preferences, if any bundle is ranked higher than it meet with any other bundle, then the join of the two is ranked higher than the latter. In other words, if x is strictly preferred to $x \wedge x'$, then x' 'brings x down' and cannot be (weakly) preferred to $x \vee x'$. If the implications are reversed, then $\succsim$ is *quasisubmodular* (see Echenique and Chambers (2008)).

It is clear that a quasisupermodular utility function represents quasisupermodular preferences and that any representation of quasisupermodular preferences must be a quasisupermodular function. Hence, quasisupermodularity of $\succsim$ is a necessary condition for $\succsim$ to admit a supermodular representation, and is a sufficient condition for quasisupermodularity of representations whenever they exist. As with real-valued functions, if $\succsim$ is monotone, the first condition in the definition of quasisupermodularity of $\succsim$ is immediate and strictly monotone preferences are both quasisupermodular and quasisubmodular.

As a cardinal property, not all representations of preferences that admit a supermodular representation need be supermodular. However, there is an extreme instance in which all representations of a preference relation in an ordinal setting are supermodular. The axiom that represents this extreme instance is called the *strong complementarity* axiom, *Axiom* (SC):

$$(SC) \quad \forall x, x \in X: \quad x' \succsim x \Rightarrow x \sim x \wedge x'$$

This axiom states that the least preferred of any two elements is indifferent to their meet. Some examples of preferences that satisfy this axiom are presented below.

Example 1. *In the case of the non-negative quadrant of* $\mathbb{R}^2$ *with the usual order* $\geq$, $(\mathbb{R}^2_+, \geq)$, *any preference relation induced by a function of the form* $f(x) = \min\{ax_1, bx_2\}$ *for some* $a, b > 0$ *satisfies Axiom* (SC).

Example 2. *Let* $X = \Delta([a,b])$ *for some real numbers* $b > a$. *Müller and Scarsini (2006) show that the partial order given by first order stochastic dominance, FOSD, renders* X *a lattice. In the lattice* $(X, FOSD)$, *the 'meet' and 'join' are given as follows. For any* $\pi, \pi' \in X$, $\pi \wedge \pi', \pi \vee \pi'$ *are the probability measures induced by the distribution functions* $F_{\pi\wedge\pi'} := \max\{F_\pi, F_{\pi'}\}$ *and* $F_{\pi\vee\pi'} := \min\{F_\pi, F_{\pi'}\}$, *respectively. Then, the preference relation* $\succsim$ *induced by* $g : X \to \mathbb{R}$ *given by* $g(\pi) = \inf(\{x \in [a,b] : F_\pi(x) = 1\})$ *satisfies Axiom* (SC). *To see this, take* $\pi, \pi' \in X$ *such that* $\pi' \succsim \pi$, *so that* $g(\pi') \geq g(\pi)$. *Take any* $x \in [a,b]$ *for which* $F_\pi(x) = 1$. *Then,* $F_{\pi\wedge\pi'(x)} = 1$ *and* $x \geq g(\pi \wedge \pi')$. *Taking infimum over such* $x \in X$ *yields* $g(\pi) \geq g(\pi \wedge \pi')$. *Conversely, take any* $x \in [a,b]$ *for which* $F_{\pi\wedge\pi'}(x) = 1$. *Then, either* $F_{\pi'}(x) = 1$ *or* $F_\pi(x) = 1$. *In the first case, by construction,* $x \geq g(\pi')$. *Taking infimum gives* $g(\pi \wedge \pi') \geq g(\pi') \geq g(\pi)$. *Similarly, if* $F_\pi(x) = 1$, *then* $x \geq g(\pi)$ *and* $g(\pi \wedge \pi') \geq g(\pi)$. *In either case, we get* $g(\pi) = g(\pi \wedge \pi')$ *and so* $\pi \sim \pi \wedge \pi'$.

The next proposition establishes that any and all representations of preferences satisfying Axiom (SC) are supermodular.

Proposition 1. *Let* $(X, \gg, \succsim)$ *be a doubly-ordered space such that* $(X, \gg)$ *is a lattice and* $\succsim$ *is a complete preorder. Assume that* $\succsim$ *satisfies Axiom* (SC). *If* $u : X \to \mathbb{R}$ *is a representation of* $\succsim$, *then* u *is non-decreasing and supermodular.*

Proof. Let $x, x' \in X$ and let $u : X \to \mathbb{R}$ be a representation of $\succsim$. Assume that there exists some $y, y' \in X$ such that $y' \gg y$ and yet $y \succ y'$. Then, $y \wedge y' = y \succ y'$ and Axiom (SC) is violated. Hence, $\succsim$ is monotone and u is non-decreasing. Then, $u(x \vee x') \geq u(x')$ and $u(x) \geq u(x \wedge x')$. By completeness of $\succsim$, either $x' \succsim x$ or $x \succsim x'$. Assume, without loss of generality, that $x' \succsim x$. Then, by Axiom (SC), $x \sim x \wedge x'$ or $u(x) = u(x \wedge x')$. Hence, $u(x \vee x') + u(x \wedge x') \geq u(x') + u(x \wedge x') = u(x') + u(x)$ and supermodularity of u follows. $\square$

Of course, Axiom (SC) is too restrictive. It makes all representations of a preference relation supermodular, while supermodularity is only a cardinal property. Moreover, many standard examples of interest, like Cobb-Douglas preferences on $(\mathbb{R}^2_+, \geq)$, do not satisfy this axiom.[9]

In Proposition 1 there is no restriction on X beyond what is absolutely necessary to be able to talk about supermodularity. A direction to turn to in looking for 'richer' supermodular representations is to lift some of the restrictions on $\succsim$ and instead impose more structure on X. The representation results in Echenique and Chambers (2008), Echenique and Chambers (2009) and Shirai (2010) can be regarded as pointing in the latter direction.

To present the logic behind their results, some additional notation and properties of $\succsim$ have to be introduced. For each $x \in X$, define $I_{\succsim}(x)$ as the indifference class of x under $\succsim$, $I_{\succsim}(x) := \{y \in X : x \sim y\}$. Let $\mathcal{I}_{\succsim}(X)$ denote the collection of indifference classes, $\mathcal{I}_{\succsim}(X) := \{I_{\succsim}(x) : x \in X\}$ and define $\succsim^*$ be the binary relation on $\mathcal{I}_{\succsim}(X)$ given by $I_{\succsim}(x') \succsim^* I_{\succsim}(x)$ if $x' \succsim x$. As $\sim$ is an *equivalence relation*[10] on X, $\mathcal{I}_{\succsim}(X)$ is the *quotient space of X with respect to $\succsim$*, which conforms a partition of X. Thus, $\succsim^*$ is well-defined, in the sense that the ranking under $\succsim^*$ of any two $I, I' \in \mathcal{I}_{\succsim}(X)$ is independent of the specific choice of members of these classes.[11] Moreover, it is a *total or complete order*[12] on $\mathcal{I}_{\succsim}(X)$, as it inherits completeness and transitivity from $\succsim$ but in addition it is antisymmetric.

Define X to be *well-ordered by* $\succsim$ if $(\mathcal{I}_{\succsim}(X), \succsim^*)$ is well-ordered, that is to say, if every subset of $\mathcal{I}_{\succsim}(X)$ contains its infimum under $\succsim^*$. If X is well ordered by $\succsim$, then we can 'line-up' all of the indifference curves according to $\succsim^*$.

Proposition 2. *Let $(X, \gg, \succsim)$ be a doubly-ordered space such that $(X, \gg)$ is a lattice and $\succsim$ is a complete preorder. If $\succsim$ is monotone and quasisupermodular, $\mathcal{I}_{\succsim}(X)$ is countable and X is well-ordered by $\succsim$, then $\succsim$ admits a non-decreasing and supermodular utility representation.*

Proof. By countability, the elements in $\mathcal{I}_{\succsim}(X)$ can be enumerated as $\mathcal{I}_{\succsim}(X) = \{I_n : n \in \mathbb{N}\}$.

[9] Cobb-Douglas preferences are those induced by functions $h : X \to \mathbb{R}$ of the form $h(x) = x_1^{\alpha} x_2^{1-\alpha}$ for some $\alpha \in [0, 1]$. Consider the case $\alpha = 1/2$ and take $x = (9, 1)$ and $x' = (1, 9)$. Then, $x \wedge x' = (1, 1)$ and $h(x') = h(x) = 3 > 1 = h(x \vee x')$, so Axiom ($SC$) is violated.

[10] An equivalence relation is a reflexive, transitive and symmetric binary relation.

[11] For any $I, I' \in \mathcal{I}_{\succsim}(X)$ and any $y \in I, y' \in I'$, $I' \succsim^* I$ if and only if $y' \succsim y$.

[12] A total or complete order is a complete, reflexive, transitive and antisymmetric binary relation.

As $\succsim^*$ is, in particular, a complete preorder, the function $d : X \to [0,1]$ given by $d(I_n) = (1/2)^n$ produces the following standard representation $\rho^* : \mathcal{I}(X) \to [0,1]$ of $\succsim^*$ on $\mathcal{I}(X)$, given by:

$$\rho^*(I) = \sum_{I' \in L_{\succsim^*}(I)} d(I')$$

Define $\rho : X \to [0,1]$ as $\rho(x) := \rho^*(I(x))$. It follows by construction of $\succsim^*$ that ρ is well-defined and represents $\succsim$. Thus, it is non-decreasing and quasisupermodular.

The set $\rho(X)$ is a countable subset of $[0,1]$, so its elements can also be enumerated. In fact, by well-ordering, they can be given an enumeration that is consistent with $\succsim$, in the sense that $\rho(X) = \{\rho_k : k \in \mathbb{N}\}$ where $\rho_{k+1} > \rho_k$. In this case, as Echenique and Chambers (2009) show, the function $u := g \circ \rho$ where $g : \rho(X) \to \mathbb{R}$ is given by $g(\rho_k) = 2^{k-1}$ represents $\succsim$ and is supermodular. $\square$

In Echenique and Chambers (2008) and Echenique and Chambers (2009), X is assumed to be finite. In this case, $\mathcal{I}_{\succsim}(X)$ is finite and X is well-ordered under any $\succsim$. Thus, if X is finite, then the conditions in Proposition 2 are very mild.[13] In this sense, Proposition 2 can be seen as the opposite extreme of Proposition 1: virtually no restrictions on $\succsim$ beyond what is necessary but severe restrictions on X. In Shirai (2010), X is a countable subset in $\mathbb{N}^n$ and so the same is true.[14]

Since strictly monotone preferences are quasisupermodular, Echenique and Chambers (2009) conclude that the assumption of supermodularity on a representation of strictly monotone preferences on a finite lattice is innocuous (or vacuous). The same is true for submodularity, as strictly monotone preferences are also quasisubmodular (Echenique and Chambers (2008), Theorem 5). Hence, preferences on finite lattices induced by strictly increasing and submodular functions admit a supermodular representation, and those induced by strictly increasing supermodular functions admit a submodular representation.

The following example from Shmaya (????) shows that, in moving to infinite lattices, monotonicity and quasisupermodularity are no longer enough to produce a supermodular

[13]Quasisupermodularity is necessary and monotonicity is a very standard assumption.

[14]Both Echenique and Chambers (2008) and Shirai (2010) present their results for submodularity instead.

representation. Hence, supermodularity has stronger implications that its ordinal counterpart.

Example 3 (Shmaya (????))**.** *Consider the lattice* $([0,1]^2, \geq)$ *and preferences induced by* $u(x) = \max\{\frac{2x_1+x_2}{3}, \frac{x_1+2x_2}{3}\}$*. These preferences are strictly monotone, hence monotone and quasisupermodular. Assume that there exists some strictly increasing* f *such that* $f \circ u$ *is supermodular. Shmaya (????) shows that* f *must be continuous (everywhere). Fix* $N \in \mathbb{N}$ *and, for each* $k \in \{2, ..., N\}$*, take* $x_k = \left(\frac{k+1}{N}, \frac{k-2}{N}\right)$ *and* $x'_k = \left(\frac{k-2}{N}, \frac{k+1}{N}\right)$*. Then, we have* $u(x_k) = u(x'_k) = \frac{k}{N}$*,* $u(x_k \vee x'_k) = \frac{k-2}{N}$ *and* $u(x_k \wedge x'_k) = \frac{k+1}{N}$*. Supermodularity implies that* $f\left(\frac{k}{N}\right) - f\left(\frac{k-2}{N}\right) \leq f\left(\frac{k+1}{N}\right) - f\left(\frac{k}{N}\right)$*. From here we get that* $f\left(\frac{N-2}{N}\right) - f(0) + f\left(\frac{N-1}{N}\right) - f\left(\frac{1}{N}\right) \leq f(1) - f\left(\frac{2}{N}\right)$ *and taking limit as* $N \to +\infty$ *yields* $f(1) \leq f(0)$*. This last inequality contradicts the assumption that* f *is strictly increasing.*

In infinite lattices, we conclude that not all quasisupermodular and non-decreasing functions are supermodularizable. However, the following result from Li Calzi (1990) shows that the non-smoothness of the utility function in the example is key.

Proposition 3. *Let* $X \subseteq \mathbb{R}^N_+$ *be a sublattice that is a direct product of compact intervals and let* $\succsim$ *be a complete preorder on* X*. Assume that* $\succsim$ *admits a representation that is twice continuously differentiable on an open superset of* X *and has positive partial derivatives on* X*. Then,* $\succsim$ *admits a supermodular utility representation.*

Proof. Let u be a representation with the stated properties. Then, as Corollary 20 in Li Calzi (1990) shows, there exists some number $r \in \mathbb{R}$ such that the composition $g \circ u$ with $g : \mathbb{R} \to \mathbb{R}$ given by $g(s) := \text{sign}(r)e^{rs}$ is supermodular. As g is a strictly increasing function, it follows immediately that $g \circ u$ also represents $\succsim$. $\square$

Utility representations that satisfy the assumptions of the proposition are strictly increasing. Thus, $\succsim$ is strictly monotone, hence quasisupermodular. As noted above, the proposition implies that preferences induced by strictly increasing submodular functions can admit supermodular representations.[15]

These facts should not be surprising in the light of Milgrom and Shannon (1994). Their Monotonicity Theorem establishes that quasisupermodularity and the single crossing crossing property characterize the desired monotone comparative static predictions. As noted, strictly increasing functions are quasisupermodular. Moreover, Corollary 20 in Li Calzi (1990) produces the supermodular function as a strictly increasing transformation of the original function, so monotonicity and single crossing properties are preserved. In this sense, the conclusion in Echenique and Chambers (2009) extends to certain non-finite cases as well: supermodularity has no content beyond quasisupermodularity for some strictly monotone 'smooth' preferences.

Revisiting Kreps (1979)

Within the realm of finite lattices, we also have the following counterpart of Propositions 1 and 4 in Echenique and Chambers (2008). Consider the following *Axiom* (GK^*):

$$(GK^*) \quad \forall x, x', y \in X : \quad x \sim x \wedge x' \Rightarrow x \wedge y \sim x \wedge x' \wedge y$$

The name (GK^*) comes from the fact that the axiom can be seen as a 'dual' of Property (GK) in Epstein and Marinacci (2007), which is in turn an extension to arbitrary lattices of Property (1.5) in Kreps (1979).[16] This axiom states that if some bundle is indifferenct to its 'meet' with another bundle, then taking the 'meet' of both with any third bundle will not alter the ranking. In the context of Kreps (1979), the axiom reads that if a subset is ranked for what it shares with a second subset, then any 'further shrinking' of the set will also be

[15] While the finite case works just as well for submodularity, the sufficient conditions for producing a g as in the proof of Proposition 3 that renders $g \circ u$ submodular are different from those in Proposition 3 as presented in Li Calzi and Veinott (2005). In particular, the inner product of the gradient is assumed to be non-positive off the diagonal, so u is not strictly increasing. Whether a different monotonicity assumption or can give both supermodularity and submodularity, as well as how much the smoothness assumptions can be relaxed, remains an open question.

[16] Axiom (GK) states that, for any $x, x', y \in X$, $x \sim x \vee x' \Rightarrow x \vee y \sim x \vee x' \vee y$.

ranked for what this 'shrinking' shares with the second subset.

Axiom (GK^*) is a weakening of axiom (SC), as the next claim shows.

Claim 1. *Let $(X, \gg, \succsim)$ be a doubly-ordered space such that $(X, \gg)$ is a lattice and $\succsim$ is a complete preorder. If $\succsim$ satisfies Axiom (SC), then it satisfies Axiom (GK^*).*

Proof. The same argument in Proposition 1 establishes that $\succsim$ is monotone. Let $x, x' \in X$ such that $x \sim x \wedge x'$ and take any $y \in X$. By monotonicity, $x \wedge x' \succsim x \succsim x \wedge y$. Axiom (SC) then gives $x \wedge y \sim x \wedge x' \wedge x \wedge y = x \wedge x' \wedge y$, as desired. □

The next claim is a counterpart of one the equivalence results in Echenique and Chambers (2008) for quasisupermodularity. It shows that, for monotone preferences, property (GK^*) is an alternative characterization of quasisupermodularity.

Claim 2. *Let $(X, \gg, \succsim)$ be a doubly-ordered space such that $(X, \gg)$ is a lattice and $\succsim$ is a monotone complete preorder. Then, $\succsim$ is quasisupermodular if and only if it satisfies (GK^*).*

Proof. Assume that $\succsim$ is monotone and satisfies (GK^*). By monotonicity, we need only worry about strict preference. Let $x, x' \in X : x \succ x \wedge x'$ and assume that, contrary to quasisupermodularity, $x' \sim x \vee x'$.[17] Since $x \vee x' \gg x$ and $x \vee x' \gg x$, we have that $(x \vee x') \wedge x' = x'$ and $(x \vee x') \wedge x = x$. Thus, by (GK^*), we have:

$$x = (x \vee x') \wedge x \sim (x \vee x') \wedge x' \wedge x = x \wedge x'$$

This contradicts the assumption that $x \succ x \wedge x'$.

Conversely, assume that $\succsim$ is monotone and quasisupermodular. Let $x, x' \in X$ such that $x \sim x \wedge x'$ and take any $y \in X$. By definition, since $x \gg x \wedge x'$ and $x \gg x \wedge y$, we have that $x \gg (x \wedge x') \vee (x \wedge y)$ and so $x \succsim (x \wedge x') \vee (x \wedge y)$ by monotonicity. Also by monotonicity, as $(x \wedge x') \vee (x \wedge y) \gg x \wedge x'$, we have that $(x \wedge x') \vee (x \wedge y) \succsim x \wedge x'$. Thus,

[17] Monotonicity rules out $x' \succ x \vee x'$.

$$x \succsim (x \wedge x') \vee (x \wedge y) \succsim x \wedge x' \sim x \Rightarrow x \wedge x' \sim (x \wedge x') \vee (x \wedge y)$$

By monotonicity, $x \wedge y \succsim (x \wedge x') \wedge (x \wedge y)$. Assume that we have strict preference. Then, quasisupermodularity implies $(x \wedge x') \vee (x \wedge y) \succ x \wedge x'$, which is absurd. Thus, GK^* is established. □

By a straightforward alteration of the arguments in Kreps (1979), quasisupermodularity and monotonicity give the following additive representation structure in the context of a finite lattice.[18]

Claim 3. *Let $(X, \gg, \succsim)$ be a doubly-ordered space such that $(X, \gg)$ is a non-empty finite lattice and $\succsim$ is a complete preorder. If $\succsim$ is monotone and quasisupermodular, there exists a set $S \subseteq X$ and a function $u : S \to \mathbb{R}_{++}$ such that $U : X \to \mathbb{R}$ defined by:*

$$U(x) := \sum_{s \in S \cap L_{\gg}(x)} u(s)$$

represents $\succsim$.

The set S in the claim can be taken to be the set of elements $s \in X$ for which, for any $x \in X$, if s is ranked indifferent to $s \wedge x$, then $x \gg s$; hence, s is the 'minimal' element in X (under $\gg$) for which it is ranked 'as high as' its meet with any other $x \in X$. In the lattice of subsets of a set under set inclusion, subset A is in S if for any subset B such that A is 'worth' only what it shares with B, then B in fact contains A. Thus, there is no 'strictly smaller' (that is to say, no proper) subset of A that represents A's 'worth' under $\succsim$; removing any further elements from A would make the resulting set strictly worse. The claim says that, under monotonicity and quasisupermodularity, the preference ranking on a finite lattice can be determined from a ranking of these 'minimal' elements, selecting, for each $x \in X$, those that rank below x under $\gg$.

[18]Finiteness plays a role in the construction of the function f and the set S in the theorem, and ensures that the sum in the representation is well-defined. The details are in the appendix.

Another interpretation of this representation is explored in the next proposition.

Proposition 4. *Let $(X, \gg, \succsim)$ be a doubly-ordered space such that $(X, \gg)$ is a non-empty finite lattice and $\succsim$ is a complete preorder. If $\succsim$ is monotone and quasisupermodular, there exists a set $S \subseteq X$ and a function $V : X \times S \to \mathbb{R}_+$ such that $U : X \to \mathbb{R}$ defined by:*

$$U(x) := \sum_{s \in S} \max_{y \in \cap L_{\gg}(x)} V(y, s)$$

represents $\succsim$.

Proof. Let $u : S \to \mathbb{R}_{++}$ be the function produced in Claim 3 and define $V : X \times S \to \mathbb{R}_+$ as $V(x, s) := u(s)I(x \gg s)$. Let $y \in L_{\gg}(x)$. Then, $y \gg s$ implies $x \gg s$; equivalently, $I(y \gg s) \leq I(x \gg s)$. Thus,

$$u(s)I(x \gg s) = \max\left(\{u(s)I(y \gg s) : y \in L_{\gg}(x)\}\right)$$

and

$$\sum_{s \in S} \max_{y \in \cap L_{\gg}(x)} V(y, s) = \sum_{s \in S} u(s)I(x \gg s) = \sum_{s \in S \cap L_{\gg}(x)} u(s)$$

By Claim 3, the latter is a representation of $\succsim$. □

If we think of the set S as a set of 'states', then the representation is an additive-separability-across-states structure. While the function $V(\cdot, s)$ need not be a representation of $\succsim$, it is the case that $V(y', s) \geq V(y, s)$ for all $s \in S$ implies that $y' \succsim y$.

3 Supermodularity in Cardinal Frameworks

The framework of choice over lotteries seems the more natural environment for cardinal properties like supermodularity. Moreover, it has several features that contribute to producing a supermodular representation. First, the primitive term in the analysis is a preference relation over lotteries, while the supermodular representation is of the induced preferences over outcomes. Thus, the preference relation is defined on one level and the order structure on a different level.[19] Second of all, the Mixture Space Theorem and the von Neumann and Morgenstern Theorem provide existence of representations and a great deal of structure on them, the expected utility structure. Therefore, the problem is simply to extend this structure to supermodularity. Finally, as a consequence of these theorems we also have uniqueness up to positive affine transformations. Thus, if one utility representation of $\succsim_X$ is supermodular, then all of those that provide $\succsim$ its expected utility structure will also be supermodular.

Mixture spaces and Borel probability measures

Let $\mathcal{T}$ be a topology on X such that $(X, \mathcal{T})$ is a T1 space, that is, a space in which all singletons are closed sets. Denote by $\mathcal{B}(X)$ the *Borel σ-field on X* and let $\Delta(X)$ denote the space of *Borel probability measures* on X, that is, the space of probability measures defined on $(X, \mathcal{B}(X))$.[20] Finally, let $\Delta^0(X) \subseteq \Delta(X)$ be the subset of *simple* Borel probability measures on X, that is, the probability measures in $\Delta(X)$ that have finite support.[21] In particular, for any $x \in X$, the point mass concentrated at x, δ_x, is a simple (Borel) probability measure.[22]

Following Fishburn (1982), a pair $(Z, *)$ where Z is a set and $* : [0,1] \times Z \times Z \to Z$ is an operation such that:

- $\forall z, z' \in Z$, $*(1, z, z') = z$
- $\forall z, z' \in Z$, $\forall \alpha \in [0,1]$, $*(\alpha, z, z') = *(1-\alpha, z', z)$

[19] The reason for this working in two separate levels is twofold. From an applied point of view, it follows the standard practice in game theoretic applications. From a technical point of view, talking about supermodularity of functions on $\Delta(X)$ requires us to specify orders that render $\Delta(X)$ a lattice. Müller and Scarsini (2006) present some examples, but not all typical orders form lattices on $\Delta(X)$ for typical sets X.

[20] The Borel σ-field is the σ-field generated by the closed subsets of X. The assumption that $(X, \mathcal{T})$ is T1 implies then that all singletons are measurable: $\{\{x\} : x \in X\} \subseteq \mathcal{B}(X)$.

[21] Alternatively, simple (Borel) probability measures are those $\mu \in \Delta(X)$ for which there exists a non-negative real-valued function $f_\mu : X \to \mathbb{R}_+$, called a *simple density function*, that has finite support and satisfies $\sum_{x \in \text{supp}(f_\mu)} f_\mu(x) = 1$ and $\mu(S) = \sum_{x \in \text{supp}(f_\mu) \cap S} f_\mu(x)$ for any Borel subset $S \subseteq X$.

[22] For any $A \in \mathcal{B}(X)$, $\delta_x(A) := I(x \in A)$, where I is the indicator function.

- $\forall z, z' \in Z$, $\forall \alpha, \beta \in [0,1]$,$*(\alpha, *(\beta, z, z'), z') = *(\alpha\beta, z, z')$

is a *mixture space*. Both $\Delta(X)$ and $\Delta^0(X)$, coupled with the standard operation of taking convex combinations of probability measures, $*(\alpha, \mu, \mu') = \alpha\mu + (1-\alpha)\mu'$, form mixture spaces.[23] Henceforth, $*$ will denote this specific mixture operation.

Let $\mathcal{D} \subseteq \Delta(X)$ be a subset of probability measures that contains all simple probability measures and forms a mixture space on its own; that is to say, $\Delta^0(X) \subseteq \mathcal{D}$ and $(\mathcal{D}, *)$ is a mixture space.[24] While $\mathcal{D}$ does not form a linear space, its mixture structure makes it possible to define a natural specialization of the notion of linearity. A function $f : \mathcal{D} \to \mathbb{R}$ is *linear in* $*$ if for all $\mu, \mu' \in \mathcal{D}$ and all $\alpha \in [0,1]$, $f\left(*(\alpha, \mu, \mu')\right) = \alpha f(\mu) + (1-\alpha) f(\mu')$.[25] Of course, any linear space is a mixture space and any linear functional is linear in $*$, with $*$ defined as the standard operation of taking convex combinations in linear spaces.

Preferences over lotteries and supermodular expected utility

Let $\succsim \subseteq \mathcal{D} \times \mathcal{D}$ be now a complete preorder on $\mathcal{D}$. Since $\mathcal{D}$ contains all point masses, $\succsim$ induces a natural complete preorder on X, called the *induced preorder* $\succsim_X$ and given by:

$$x' \succsim_X x \quad \text{if} \quad \delta_{x'} \succsim \delta_x$$

for any $x, x' \in X$. It is easy to see that the asymmetric and symmetric parts of $\succsim_X$, denoted by $\succ_X, \sim_X$ respectively, are the binary relations induced by the asymmetric and symmetric part of $\succsim$. The focus is on the connection between supermodularity of representations of $\succsim_X$ and properties of $\succsim$. In game theoretic and other cardinal utility applications, supermodularity is imposed on the Bernoulli utility function, which represents $\succsim_X$. However, the primitive preference relation is $\succsim$. Thus, the relevant link is the link between supermodularity of representations of $\succsim_X$ and properties of $\succsim$.

[23]The support of the mixture of two probability measures is the union of the supports if the mixture has $\alpha \in (0,1)$; otherwise, it coincides with the support of the distribution with coefficient 1 in the mixture.

[24]The operation $*$ in $(\mathcal{D}, *)$ is of course the restriction of $*$ to $\mathcal{D} \times \mathcal{D}$.

[25]Fishburn (1982) refers to this property simply as linearity. The use of the modifier 'in $*$' is taken from Kreps (Forthcoming).

In the *Mixture Space Theorem* (Herstein and Milnor (1953)), the following three axioms are imposed on $\succsim$:[26]

(a) $\succsim$ is a complete preorder

(b) $\forall \mu, \mu', \mu'' \in \mathcal{D}, \forall \alpha \in (0,1): \quad \mu' \succ \mu \Rightarrow *(\alpha, \mu', \mu'') \succ *(\alpha, \mu, \mu'')$

(c) $\forall \mu, \mu', \mu'' \in \mathcal{D}: \mu \succ \mu' \succ \mu'', \exists \alpha, \beta \in (0,1): *(\alpha, \mu, \mu'') \succ \mu' \succ *(\beta, \mu, \mu'')$

Axiom (a) is a basic assumption on $\succsim$, needed for $\succsim$ to admit a numerical representation. Axiom (b) is an *independence* assumption, stating that the presence of a third lottery μ'' does not change the ranking of μ, μ' when mixed with 'equal weight'. Finally, Axiom (c) is a *continuity* or *Archimedean* axiom. Following Kreps (Forthcoming), this last axiom rules out the existence of 'supergood' or 'superbad' lotteries: no matter how high μ is ranked by the consumer, for some mixture, μ' is still strictly preferred to this mixture of μ and μ''. Similarly for μ'': no matter how low it is ranked, μ' is still strictly worse than some mixture of μ and μ''.

The Mixture Space Theorem states that a binary relation $\succsim$ on $\mathcal{D}$ satisfies these three axioms if and only if there exists a real-valued function u on $\mathcal{D}$ representing $\succsim$ that is linear in $*$ and unique up to positive affine transformations. That is, any other function on $\mathcal{D}$ that represents $\succsim$ and is is linear in $*$ is a positive affine transformation of u. Thus, being a 'linear-in-$*$' representation of $\succsim$ is a cardinal property.

If $\mathcal{D} = \Delta^0(X)$, the *von Neumann and Morgenstern Expected Utility Representation Theorem* (henceforth, the von Neumann and Morgenstern Theorem; von Neumann and Morgenstern (1953)) establishes the existence of a real-valued function U on X that is also unique up to positive affine transformations and such that $u(\mu) = \int U d\mu$. The case $\mathcal{D} = \Delta(X)$ can also be handled if there exists a metric d on X such that (X, d) is separable and if $\succsim$ is appropriately continuous.[27] In either case, it is clear that the function U in the von

[26] In Fishburn (1982) and Kreps (1988) Axiom (a) is stated with $\succ$ as the primitive term. Kreps (Forthcoming) presents a different set of axioms that are equivalent to $(a), (b), (c)$ here.

[27] In any metric space, the metric topology is T1 and so there is no conflict in endowing X with a metric and working with its metric topology. The 'appropriate' notion of continuity is specified later on. For more details, see Kreps (1988), Kreps (Forthcoming).

Neumann and Morgenstern Theorem represents $\succsim_X$, as $U(x) = u(\delta_x)$. Hence, the problem is to establish the link between properties of $\succsim$ and supermodularity of U.

The obvious link is given by the following axiom, which will be called *Axiom* (S):

$$(S) \quad \forall x, x \in X: \quad *\left(\frac{1}{2}, \delta_{x\wedge x'}, \delta_{x\vee x'}\right) \succsim *\left(\frac{1}{2}, \delta_x, \delta_{x'}\right)$$

Axiom (S) states that, for any two outcomes, the 50-50 mixture between the 'highest' and the 'lowest' of the two (under $\gg$) is weakly preferred to the 50-50 mixture between the outcomes.[28] If we think of X as the product of two lattices with the product order, the axiom can be read as saying that a 50-50 mixture between 'all high' coordinates or 'all-low' coordinates is weakly preferred to a 50-50 lottery between elements that feature both high and low coordinates. Thus, it can be read as a 'complementarity across dimensions' axiom.

Proposition 5. *Let* $(X, \gg)$ *be a lattice and let* $\succsim$ *be a binary relation on* $\Delta^0(X)$*. Then,* $\succsim$ *satisfies axioms* $(a), (b), (c)$ *and* (S) *if and only if there exists a supermodular real-valued function* $U : X \to \mathbb{R}$ *such that* $u : \Delta^0(X) \to \mathbb{R}$ *given by* $u(\mu) = \sum_{x\in supp(\mu)} U(x)\mu(\{x\})$ *represents* $\succsim$*. Moreover,* U *is unique up to positive affine transformations.*

Proof. Assume that preferences $\succsim$ are represented by $u(\mu) = \sum_{x\in\text{supp}(\mu)} U(x)\mu(\{x\})$ for some real-valued supermodular function U. Then, u is linear in $*$. That $\succsim$ satisfies axioms $(a), (b)$ and (c), follows thus from the Mixture Space Theorem. To verify that $\succsim$ satisfies (S), take any $x, x' \in X$. Using linearity in $*$ of u and supermodularity of U, we find that:

$$\begin{aligned} u\left(*\left(\frac{1}{2}, \delta_{x\wedge x'}, \delta_{x\vee x'}\right)\right) = \frac{1}{2}u(\delta_{x\wedge x'}) + \frac{1}{2}u(\delta_{x\vee x'}) &= \frac{1}{2}U(x\wedge x') + \frac{1}{2}U(x\vee x') \\ &\geq \frac{1}{2}U(x) + \frac{1}{2}U(x') \\ &= \frac{1}{2}u(\delta_x) + \frac{1}{2}u(\delta_{x'}) = u\left(*\left(\frac{1}{2}, \delta_x, \delta_{x'}\right)\right) \end{aligned}$$

[28] Of course, the property is trivial if x, x' are comparable under $\gg$, in which case $x \vee x', x \wedge x' \in \{x, x'\}$.

Thus, Axiom (S) follows.

Conversely, assume that preferences satisfy axioms $(a), (b), (c)$ and (S). The Mixture Space Theorem produces a real-valued function $u : \Delta^0(X) \to \mathbb{R}$ that represents $\succsim$ and is linear in $*$. The von Neumann and Morgenstern Theorem produces a function $U : X \to \mathbb{R}$ such that $u(\mu) = \sum_{x \in \text{supp}(\mu)} U(x)\mu(\{x\})$. Clearly, U represents $\succsim_X$. Supermodularity of U is a simple consequence of (S) and linearity of u:

$$\begin{aligned} u(\delta_{x\wedge x'}) + u(\delta_{x\vee x'}) = 2\left(\frac{1}{2}u(\delta_{x\wedge x'}) + \frac{1}{2}u(\delta_{x\vee x'})\right) &= 2u\left(*\left(\frac{1}{2}, \delta_{x\wedge x'}, \delta_{x\vee x'}\right)\right) \\ &\geq 2u\left(*\left(\frac{1}{2}, \delta_x, \delta_{x'}\right)\right) \\ &= 2\left(\frac{1}{2}u(\delta_x) + \frac{1}{2}u(\delta_{x'})\right) = u(\delta_x) + u(\delta_{x'}) \end{aligned}$$

and thus $U(x \vee x') + U(x \wedge x') = u(\delta_{x\vee x'}) + u(\delta_{x\wedge x'}) \geq u(\delta_x) + u(\delta_{x'}) = U(x) + U(x')$ for any two $x, x' \in X$, as desired.

The statement about uniqueness up to positive affine transformations follows from the von Neumann and Morgenstern Theorem. This completes the proof. □

For the extension of the proposition to $\Delta(X)$, let $\mathcal{T}_\Delta$ be a topology on $\Delta(X)$. We say that $\succsim$ *is continuous (in* $\mathcal{T}_\Delta$*)* if, for all $\mu \in \Delta(X)$, the sets $\{\mu' \in \Delta(X) : \mu' \succ \mu\}$ and $\{\mu' \in \Delta(X) : \mu \succ \mu'\}$ are in $\mathcal{T}_\Delta$. When the space in question is a metric space, the natural topology is the metric topology.

Proposition (Extension of Proposition 5 to $\Delta(X)$)**.** *Let* $(X, \gg)$ *be a lattice and let* $\succsim$ *be a binary relation on* $\Delta(X)$*. Let* d *be a metric on* X *such that* (X, d) *is separable and let* $\Delta(X)$ *be endowed with the Prokhorov metric. Then,* $\succsim$ *satisfies axioms* $(a), (b), (c), (S)$ *and is continuous if and only if there exists a continuous, bounded and supermodular real-valued function* $U : X \to \mathbb{R}$ *such that* $u : \mathcal{D} \to \mathbb{R}$ *given by* $u(\mu) = \int U d\mu$[29] *represents* $\succsim$*. Moreover,* U *is unique up to positive affine transformations.*

[29] As a continuous function, U is Borel-measurable. Thus, the proposed characterization of u is well-defined.

The 50-50 mixture specified by Axiom (S) is crucial in the proof of Proposition 5. For other mixtures, quasisupermodularity follows instead. Consider the following weaker version of Axiom (S), called *Axiom* (qS):

$$(qS) \quad \forall x, x' \in X: \quad \exists \alpha \in (0,1) : *(\alpha, \delta_{x\wedge x'}, \delta_{x\vee x'}) \succsim *(\alpha, \delta_x, \delta_{x'})$$

Axiom (qS) states that, for any two outcomes, there exists a (strict) mixture between the 'highest' and the 'lowest' of the two that is weakly preferred to the same mixture between the outcomes themselves.[30] However, this mixture may be different that $1/2$ and need not be uniform across X, it may depend on the choice of $x, x' \in X$.[31]

Additionally, consider the following axiom of monotonicity in outcomes, *Axiom* (M):

$$(M) \quad \forall x, x' \in X: \quad x' \gg x \Rightarrow \delta_{x'} \succsim \delta_x$$

Coupling axioms (qS) and (M) yield the following counterparts of Proposition 5.

Proposition 6. *Let $(X, \gg)$ be a lattice and let $\succsim$ be a binary relation on $\Delta^0(X)$. Then, $\succsim$ satisfies axioms $(a), (b), (c), (qS)$ and (M) if and only if there exists a non-decreasing and quasisupermodular real-valued function $U : X \to \mathbb{R}$ such that $u : \Delta^0(X) \to \mathbb{R}$ given by $u(\mu) = \sum_{x\in supp(\mu)} U(x)\mu(\{x\})$ represents $\succsim$. Moreover, U is unique up to positive affine transformations.*

Proof. Assume that preferences $\succsim$ are represented by $u(\mu) = \sum_{x\in \text{supp}(\mu)} U(x)\mu(\{x\})$ for some non-decreasing, quasisupermodular real-valued function U. As before, axioms $(a), (b), (c)$ are consequences of the Mixture Space Theorem. Moreover, that $\succsim$ satisfies Axiom (M) is immediate from U being non-decreasing: for any $x, x' \in X$, $x' \gg x \Rightarrow u(\delta_{x'}) = U(x') \geq$

[30] Again, the property is trivial if x, x' are comparable under $\gg$.

[31] If the mixture in Axiom (qS) is uniform across x, then representations will satisfy the following property, weaker than supermodularity but stronger than quasisupermodularity. A function $f : X \to \mathbb{R}$ defined on a lattice $(X, \gg)$ is α-*supermodular* if there exists some $\alpha \in [0,1]$ such that, for all $x, x' \in X$, $\alpha f(x \wedge x') + (1 - \alpha)f(x \vee x') \geq \max\{\alpha f(x) + (1-\alpha)f(x'), \alpha f(x') + (1-\alpha)f(x)\}$.

$U(x) = u(\delta_x)$. It only remains to verify that $\succsim$ satisfies (qS). To this end, take any $x, x' \in X$. By monotonicity, $U(x) \geq U(x \wedge x')$. If this inequality is strict, then quasisupermodularity implies that $U(x \vee x') > U(x')$. In this case, define the function $h : [0,1] \to \mathbb{R}$ as

$$h(\alpha) = (1-\alpha)[U(x \vee x') - U(x')] + \alpha[U(x \wedge x') - U(x)]$$

This function is clearly continuous and satisfies $h(0) > 0$ and $h(1) < 0$. Thus, there exists some $\alpha^* \in (0,1)$ close enough to 0 such that $h(\alpha^*) > 0$, which implies:

$$\begin{aligned} u\left(*\left(\alpha^*, \delta_{x \wedge x'}, \delta_{x \vee x'}\right)\right) = \alpha^* u(\delta_{x \wedge x'}) + (1-\alpha^*) u(\delta_{x \vee x'}) &= \alpha^* U(x \wedge x') + (1-\alpha^*) U(x \vee x') \\ &> \alpha^* U(x) + (1-\alpha^*) U(x') \\ &= \alpha^* u(\delta_x) + (1-\alpha^*) u(\delta_{x'}) \\ &= u\left(*\left(\alpha^*, \delta_x, \delta_{x'}\right)\right) \end{aligned}$$

If $U(x) = U(x \wedge x')$ instead, by monotonicity, $U(x \vee x') \geq U(x')$ and so $U(x \vee x') + U(x \wedge x') \geq U(x') + U(x \wedge x') = U(x') + U(x)$. Thus, in this case, we can take $\alpha = \frac{1}{2}$ as in Proposition 5.

Conversely, assume that preferences satisfy axioms $(a), (b), (c), (qS)$ and (M). Let U on X be as in the proof of Proposition 5. By (M), U is non-decreasing. Quasisupermodularity of U is an immediate consequence of (qS). Take any two $x, x' \in X$. Then, there exists some $\alpha \in (0,1)$ such that:

$$\begin{aligned} \alpha U(x \vee x') + (1-\alpha) U(x \wedge x') &\geq \alpha U(x) + (1-\alpha) U(x') \\ \alpha[U(x \vee x') - U(x)] &\geq (1-\alpha)[U(x') - U(x \wedge x')] \end{aligned}$$

and so $U(x') > U(x \wedge x')$ implies $U(x \vee x') > U(x)$. Thus, U is quasisupermodular.

Again, the last statement in the proposition follows from the von Neumann and Morgenstern Theorem, completing the proof. □

Proposition (Extension of Proposition 6 to $\Delta(X)$)**.** *Let $(X, \gg)$ be a lattice and let $\succsim$ be a binary relation on $\Delta(X)$. Let d be a metric on X such that (X, d) is separable and let $\Delta(X)$ be endowed with the Prokhorov metric. Then, $\succsim$ satisfies axioms $(a), (b), (c), (qS)$ and (M) and is continuous if and only if there exists a continuous, bounded, non-decreasing and quasisupermodular real-valued function $U : X \to \mathbb{R}$ such that $u : \Delta(X) \to \mathbb{R}$ given by $u(\mu) = \int U d\mu$ represents $\succsim$. Moreover, U is unique up to positive affine transformation.*

In this sense, supermodularity is a knife-edge property. For different weights, quasisupermodularity follows instead. Now Axiom (M) is clearly responsible for U being non-decreasing. However, it also plays an important role in the proof of quasisupermodularity of U, as in Echenique and Chambers (2009). The following example shows that the assumption of U being non-decreasing cannot be dispensed with in going in the other direction.

Example 4. *Let $X = \{x, x', x \wedge x', x \vee x'\}$, with x, x' incomparable under $\gg$ and let $\succsim$ on $\Delta(X)$ (which, for finite X, is also $\Delta^0(X)$) be induced by $u(\mu) = \sum_{y \in supp(\mu)} U(y)\mu(\{y\})$, with $U(x) = 1, U(x') = 3$ and $U(x \wedge x') = 2, U(x \vee x') = 0$. Clearly, U fails to be non-decreasing. However, it is quasisupermodular: the only problematic pair is (x, x') and for this pair we have that $U(x) < U(x \wedge x')$. Nonetheless, Axiom (qS) is violated: for any $\alpha \in (0, 1)$,*

$$\alpha U(x \vee x') + (1 - \alpha)U(x \wedge x') = 2(1 - \alpha) < \alpha + 3(1 - \alpha) = \alpha U(x) + (1 - \alpha)U(x')$$

and so no $\alpha \in (0, 1)$ as in Axiom (qS) can be produced for x, x'.

Discussion

In the context of choice under uncertainty, X is a set of 'prizes' and probability measures over X are interpreted as 'lotteries'. Consider, as an example, the case $X = \mathbb{R}^2_{++}$ endowed with the usual topology and the usual order. For any $x, x' \in \mathbb{R}^2_{++}$, the lottery $*(\frac{1}{2}, x \vee x', x \wedge x')$ is a 'mean preserving spread' *in outcomes* of lottery $*(\frac{1}{2}, x, x')$.[32] Nonetheless,

[32] They both have the same mean and the difference between their variance/covariance matrices is a positive semi-definite matrix.

there is no conflict between risk aversion and supermodularity, as there are examples of both supermodular and convex functions and supermodular and concave ones.[33] Axioms (S), (qS) are statements about 'complementarity across dimensions': 'all-high / all-low' fair lotteries are weakly preferred to the corresponding 'high-low' lotteries. The 'spread' is in the value of the outcomes according to $\gg$, not in the 'risk' associated with a lottery for a fixed mean and support.

In the context of game theory, let $I \in \mathbb{N}$ be the number of players in a game and $\mathcal{I} = \{1, ..., I\}$, the set of players. Let $\{(X_i, \gg_i) : i \in \mathcal{I}\}$ be a (finite) collection of lattices that represent the spaces of pure strategies for each player and $(X, \gg)$ is the product lattice and represents the space of pure strategy profiles. The probability measures over X represent mixed or correlated strategy profiles. For any $i \in \mathcal{I}$, fix the pure strategy profile of the opponents, x_{-i}. Then, two different pure strategies for i, $x_i, x_i' \in X_i$ identify two different pure strategy profiles $x, x' \in X$ and the join and meet of the latter are $x \vee x' = (x_i \vee_i x_i', x_{-i})$ and $x \wedge x' = (x_i \wedge_i x_i', x_{-i})$. Axiom (S) says that i weakly prefers to flip a coin between 'her highest' and 'her lowest' of the two pure strategies to flipping a coin between the strategies themselves, while (qS) states the existence of some such (strict) mixing (allowing the mixture to depend on the underlying pure strategy profiles).

Proposition 2 establishes an equivalence between supermodular and quasisupermodular representability of monotone preferences on X. Things are different in the cardinal setting. Two expected utility representations of the same preferences over lotteries are a positive affine transformations of each other, so both would be supermodular if any one is. The equivalence in ordinal settings does not allow for induced preferences having a non-decreasing and quasisupermodular representation that is not also supermodular.[34] Similarly, in cardinal contexts with strictly monotone preferences over outcomes, the assumption of supermodularity is still not vacuous as it implies a certain ranking of certain lotteries.

[33] For example, $U(x) = e^{x_1+x_2}$ is supermodular and convex, while $U(x) = (x_1x_2)^{1/2}$ is supermodular and concave. Thus, supermodularity is consistent with any attitude towards risk.

[34] Let $X = \{0,1\}^2$ under the usual order and let induced preferences be represented by $U : X \to \mathbb{R}$, where $U(0,0) = 0$, $U(1,0) = U(0,1) = 1$ and $U(1,1) = \frac{3}{2}$. This function U is non-decreasing. As for quasisupermodularity and supermodularity, the only problematic case is that of $(1,0), (0,1)$ (all other pairs are comparable under the partial order). Since $U(1,1) > U(1,0) = U(0,1) > U(0,0)$, U is quasisupermodular. However, it is not supermodular: $U(1,1) + U(0,0) = \frac{3}{2} < 2 = U(1,0) + U(1,0)$. Notwithstanding, Proposition 2 implies that these preferences do admit a supermodular utility representation.

4 Conclusion

Order-theoretic approaches have become pervasive in economics, both in choice theory and game theory. The imposition of some form of supermodular structure on representations of preferences raises the issue of what are the decision-theoretical foundations of such structure. Since supermodularity is a cardinal property, it is a more natural structure in problems of choice over probability measures under the assumptions of the von Neumann and Morgenstern Theorem. In this cardinal setting, additional axioms on the preference relation over probability measures are presented that capture the full implications of supermodularity of the representation of the induced preferences.

The problem of providing axiomatic foundations for supermodularity in ordinal settings (in which supermodularity is sometimes imposed despite its cardinal nature) is much more complicated. Some extreme cases, both extreme in terms of the structure imposed on preferences and in terms of the structure imposed on the set, are well established. It is still a matter of ongoing work to strike a balance and provide axiomatic bases for supermodularity of representations for a larger class of preferences than those in Proposition 1, defined on a larger class of lattices than those covered by Proposition 2.

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Appendix

Lemma 1 (Corollary 5, Echenique and Chambers (2009))**.** *Let $(X, \gg)$ be a lattice and $f : X \to \mathbb{R}$ be strictly increasing. Then, f is both quasisupermodular and quasisubmodular.*

Proof. By monotonicity, we need only check the 'strict' statements of quasisupermodularity and quasisubmodularity. Notice that, for any $x, x' \in X$ such that $x \neq x \wedge x'$, it must be that $x' \neq x' \vee x$. Otherwise, $x' = x' \vee x \gg x$ and we would have $x = x \wedge x'$. Similarly, $x' \neq x \vee x'$ implies that $x \neq x \wedge x'$, for $x' \gg x \wedge x' = x$ gives $x' = x \vee x'$.

Let $x, x' \in X$ such that $f(x) > f(x \wedge x')$. Then, $x \neq x \wedge x'$ and so $x \vee x' \neq x'$. Since $x \vee x' \gg x'$, f being strictly increasing implies that $f(x' \vee x) > f(x')$. This gives quasisupermodularity. Similarly, assume that $f(x \vee x') > f(x')$. Then, $x \vee x' \neq x'$ and so $x \neq x \wedge x'$. As $x \gg x \wedge x'$, $f(x) > f(x \wedge x')$ and quasisubmodularity follows. □

Lemma 2. *Let $(X, \gg, \succsim)$ be a doubly-ordered space such that $(X, \gg)$ is a non-empty finite lattice and $\succsim$ is a complete preorder. Let $\geq^* \subseteq X \times X$ be given by:*

$$x' \geq^* x \quad \textit{if} \quad x \sim x \wedge x'$$

If $\succsim$ is monotone and quasisupermodular, then:

a) $\geq^*$ is reflexive and transitive

b) $\geq^*$ is monotone

c) For any $x, x', x'' \in X$, if $x \geq^* x'$ and $x' \gg x''$, then $x \geq^* x''$

d) For any $x, x', x'', x''' \in X$, if $x' \geq^* x$ and $x''' \geq^* x''$, then $x' \wedge x''' \geq^* x \wedge x''$

e) For every $x \in X$, let $x \in A \subseteq U_{\geq^*}(x)$; then, $\wedge A \in U_{\geq^*}(x)$

f) For every $x \in X$ there exists some $x' \in X$ with $x \gg x'$ such that, for any $x'' \in X$, $x'' \geq^* x$ if and only if $x'' \gg x'$

Proof. By Claim 2, $\succsim$ satisfy (GK^*).

a) That $\geq^*$ is reflexive is immediate from the fact that $x \wedge x = x$. Let $x, x', x'' \in X$ such that $x' \geq^* x$ and $x'' \geq^* x'$. Then, $x \sim x \wedge x'$ and $x' \sim x' \wedge x''$. By (GK^*) applied to the latter, $x \wedge x' \sim x \wedge x' \wedge x''$. Then, using monotonicity,

$$x \succsim x \wedge x'' \succsim x \wedge x' \wedge x'' \sim x \wedge x' \sim x$$

which implies $x \sim x \wedge x''$ or $x'' \geq^* x$. Thus, $\geq^*$ is transitive.

b) Let $x, x' \in X$ such that $x' \gg x$. Then, $x \wedge x' = x$ and the result follows immediately.

c) By b), we have $x' \geq^* x''$. The result then follows by transitivity of $\geq^*$.

d) By definition, we have $x \sim x \wedge x'$ and $x'' \sim x''' \wedge x''$. By (GK^*), we have $x \wedge x'' \sim x \wedge x' \wedge x''$ and $x \wedge x' \wedge x'' \sim x \wedge x' \wedge x'' \wedge x'''$. Thus, by transitivity of $\succsim$, $x \wedge x'' \sim x \wedge x' \wedge x'' \wedge x'''$ or $x' \wedge x''' \geq^* x \wedge x''$.

e) Let $x \in X$ and let $x \in A \subseteq U_{\geq^*}(x)$. Such A exists by reflexivity of $\geq^*$. For $n \in \mathbb{N}$, let $\#A = n + 1$. If $n = 1$, let $x'' \in A \setminus \{x\}$ and let $x' := \wedge A = x \wedge x''$. By reflexivity of $\geq^*$ and by d), $x' = x \wedge x'' \geq^* x \wedge x = x$ and so $x' \in U_{\geq^*}(x)$. Let $k \in \mathbb{N}$ and assume that the statement is true when $n = k$. Consider the case $n = k + 1$. Let $\widetilde{x} \in A$ and define $x' := \wedge A$ and $x'' := \wedge(A \setminus \{\widetilde{x}\})$. Then, $x' := \wedge A = x'' \wedge \widetilde{x}$. By the induction hypothesis, $x'' \geq^* x$; therefore, by d), $x' = x'' \wedge \widetilde{x} \geq^* x \wedge x = x$ and so $x' \in U_{\geq^*}(x)$.

f) Let $x \in X$ and let $U_{\geq^*}(x)$ be its $\geq^*$-upper contour set. By reflexivity of $\geq^*$, $U_{\geq^*}(x)$ is non-empty. By finiteness of X, $(X, \gg)$ is a complete lattice. Let $x' := \wedge U_{\geq^*}(x)$. Then, x' is well-defined. By reflexivity of $\succsim$ and by definition of $\wedge$ we have $x \gg x'$. Moreover, by e), $x' \geq^* x$. Let $x'' \in U_{\geq^*}(x)$. Then, $x'' \gg x'$ by definition of $\wedge$. Conversely, let $x'' \in X$ such that $x'' \gg x'$. By b), $x'' \geq^* x'$ and so transitivity of $\geq^*$ implies $x'' \geq^* x$.

This completes the proof. $\square$

Let $f : X \to X$ be $f(x) := \wedge U_{\geq^*}(x)$. Then, f) can be rephrased as stating that for any $x, \widetilde{x} \in X$, $\widetilde{x} \geq^* x$ if and only if $\widetilde{x} \gg f(x)$.

Lemma 3. *Consider the same conditions as in Lemma 2 and consider the function f defined above. For any $x, x' \in X$:*

a) $f(f(x)) = f(x)$

b) $x' \geq^* x$ *if and only if* $f(x') \gg f(x)$

c) f *is monotone*

d) $x =^* f(x)$

e) $x \sim f(x)$

Proof. Let $x, x' \in X$.

a) From e) and f) in Lemma 2, we have that $f(x) \geq^* x$ and $f(f(x)) \geq^* f(x)$. Thus, by transitivity of $\geq^*$, $f(f(x)) \geq^* x$. By another use of f), $f(f(x)) \gg f(x)$. Moreover, by construction, $f(x) \gg f(f(x))$. Thus, the result follows by antisymmetry of $\gg$.

b) Assume that $x' \geq^* x$. Then, by transitivity of $\geq^*$, $f(x') \geq^* x$ and so $f(x') \gg f(x)$ by construction. Conversely, assume that $f(x') \gg f(x)$. Then, by construction and by transitivity of $\gg$, $x' \gg f(x)$. Thus, by f) in Lemma 2, it follows that $x' \geq^* x$.

c) Assume $x' \gg x$. Then, by monotonicity of $\geq^*$, $x' \geq^* x$. By part b) of the Lemma, $f(x') \gg f(x)$.

d) By construction, $x \gg f(x)$. Thus, by monotonicity of $\geq^*$, $x \geq^* f(x)$. Conversely, by part e) of Lemma 2, $f(x) \geq^* x$.

e) By monotonicity of $\succsim$, it follows from the fact that $x \gg f(x)$ that $x \succsim f(x)$. Conversely, by part d) of Lemma 2, we have that $f(x) \geq^* x$, or $x \sim x \wedge f(x)$. By monotonicity and transitivity of $\succsim$, it follows from $f(x) \gg f(x) \wedge x$ that $f(x) \succsim x$.

This completes the proof. □

Lemma 4. *Let Y be a non-empty finite set, $\succcurlyeq_1$ a complete preorder on Y and $\succcurlyeq_2$ a partial order on Y. If $y' \succ_2 y$ implies $y' \succ_1 y$ for any $y,' y \in Y$ such that $y \neq y'$, then there exists some $a \in \mathbb{R}^{\#Y}_{++}$ such that the function $U : Y \to \mathbb{R}_+$ given by:*

$$U(y) := \sum_{\tilde{y} \in L_{\succcurlyeq_2}(y)} a_{\tilde{y}}$$

represents $\succcurlyeq_1$.

Proof. Let $\mathcal{I}_1 := \{I_{\sim_1}(y) : y \in Y\}$ be the quotient space of Y over the equivalence relation induced by $\succcurlyeq_1$. By finiteness, the quotient space can be well-ordered by the natural order induced by $\succcurlyeq_1$ (see Proposition 2). Thus, we can write $\mathcal{I}_1 = \{I_i : i \in \{1, ..., n\}\}$ for some $n \in \mathbb{N}$ and such that for all $i, j \in \{1, ..., n\}$ and for all $y \in I_i, y' \in I_j$, we have that $y' \succ_1 y$ if $j > i$. Start with $i = 1$. Pick any real number $a_1 > 0$ and set $a_y := a_1 =: b_y$ for all $y \in I_1$. By the assumption on the way the two binary relations interact, for any such y, $L_{\succcurlyeq_2}(y) = \{y\}$: by reflexivity of $\succcurlyeq_2$, $y \in L_{\succcurlyeq_2}(y)$ for any $y \in Y$; if there is any $\tilde{y} \in L_{\succcurlyeq_2}(y) \setminus \{y\}$, then $y \succ_2 \tilde{y}$ and hence $y \succ_1 \tilde{y}$, which implies that $y \notin I_1$. Consider next the case $i = 2$. For any $y \in I_2$ and any $\tilde{y} \in L_{\succ_2}(y)$, by assumption, $y \succ_1 \tilde{y}$ and so $L_{\succ_2}(y) \subseteq I_1$. Thus, we can define $w_2^* : I_2 \to \mathbb{R}_+$

$$w_2^*(y) := \sum_{\tilde{y} \in L_{\succ_2}(y)} a_{\tilde{y}}$$

with $w_2^*(y) := 0$ is the index set of the sum is empty. Then,

$$w_2^*(y) = a_1 \# L_{\succ_2}(y)$$

Pick any $b_2 > \max\{b_1, \max(\{w_2^*(\hat{y}) : \hat{y} \in I_2\})\}$ and set $a_y := b_2 - w_2^*(y)$. Then, $b_2 > b_1$ and for all $y \in I_2$, $a_y > 0$ and $a_y + w_2^*(y)$ is constant on I_2.

Take now $i = k + 1$ given some $k \in \{2, ..., n - 2\}$. Assume that we have a set of positive numbers $\{a_y : y \in \cup_{i=1}^{k+2} I_i\}$ and the function $w_{k+2}^* : \cup_{i=1}^{k+2} I_k \to \mathbb{R}_+$ defined as:

$$w_{k+2}^*(y) := \sum_{\tilde{y} \in L_{\succ_2}(y)} a_{\tilde{y}}$$

(and 0 when the sum is over an empty set) such that $a_y + w_{k+2}^*(y)$ is constant on each

$I_i : i \in \{1, ..., k+2\}$ and strictly increasing across $i \in \{1, ..., k+2\}$. Take $y \in I_{k+2}$. Then, $L_{\succ_2}(y) \subseteq \cup_{i=1}^{k+1} I_i$ and so we can define w^*_{k+3} as the natural extension of w^*_{k+2}. Choose $b_{k+3} > \max\{a_{y_0} + w^*_{k+2}(y_0), \max(\{w^*_{k+3}(\hat{y}) : \hat{y} \in \cup_{i=1}^{k+1} I_i\})\}$ for any $y_0 \in I_{k+2}$. Then, let $a_y := b_{k+3} - w^*_{k+3}(y)$. Thus, $a_y > 0$ and $a_y + w^*_{k+3}(y)$ is constant on $I_i : i \in \{1, ..., k+3\}$. This establishes the desired representation, setting $U(y) := a_y + w^*_n(y)$ for any $y \in Y$. $\square$