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An efficient algorithm for the computation of a solution to von Neumann's model

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An efficient algorithm for the computation of a solution to von Neumann's model

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S U M M A R Y

The possibility of application of the von Neumann model was hampered by the non existence of an efficient algorithm for its solution. The essential difficulty lies in the fact that in general there is no solution in the rational field, solutions are related to high degree polynomial roots. Therefore, an iterative method is proposed permitting a rapid approximation to the solution by means of reiterated solutions of linear programming problems.

Using the simplex method it is possible to solve rapidly the linear program in each step, thus obtaining the maximum rate of growth by a false position method.

Since the translation of Von Neumann's paper,[2] the model there presented has received considerable attention in the literature. Much is know about its properties: the existence of proportional growth solutions under less restrictive assumptions than those of the original model, the "turnpike" property of efficient growth paths, the dual relationship between activity levels and growth rate with prices and interest rate, all have been extensively considered. Nevertheless, the possibility of its application to development planning has found a great obstacle in the difficulties of computing an actual solution. Algorithms for finding a solution in a finite number of steps are know (Weil),[3] but the computional effort is great.[1] The essential difficulty resides in the fact that in general there is no solution in the rational field. At every step it is necessary to compute all the roots of a high degree polynomial, and solve a large quantity of systems of equations. In what follows we shall give a more effective procedure, which approximates the solution efficiently.

[1] At the time of writing this note, another article[1] was unknown to me. Even though cast in the language of game theory, the algorithm presents some similarities with the one here proposed. The present version is more efficient.

We shall limit ourselves to present an algorithm for the computation of a proportional growth solution at a maximal rate. Once the so called Von Neumann ray (or facet, when the solution is not unique) is known, further computational efforts can be drastically reduced, since due to the turnpike property most of the activities used during a long time interval will be those used in the proportional growth solution.

The Von Neumann model, for the case of a finite number of basic activities, can be summarized as follows. There are m goods in the economy, which are produced by means of themselves. Production is carried out by n linear processes, each using up a certain amount of the commodities to produce certain quantities, which will be available one period later. Thus the productive process is to be understood as a transformation of given stocks of commodities into new ones, with a gestation lag of one period, including the original capital stock at a later stage of wear and tear.

All the information about the technology can be presented in two $m \times n$ matrices; the input matrix A and the output matrix B. The i^{th} row of both matrices corresponds to the i^{th} commodity, whereas the j^{th} column corresponds to the j^{th} activity; the columns of A give the cost structure, and those of B indicate how much is produced by each process operated at unit level.

If the elements of the non-negative n - vector $\underline{x}$ stand for the levels at which each activity is operated during a given period, Bx will give us the quantities which will be available at the end of the period, whereas Ax gives the amounts of commodities used up. Since we are interested in proportional growth solutions, we do not need to know the scale of the system, so that we may normalize the activity levels requiring, for simplicity, that they add up to unity; as a consequence $\underline{x}$ will be a probability vector.

Proportional growth means that one period later the activity levels $\underline{x}$ will be multiplied by some positive factor of proportionality μ (we do not exclude shrinking economies, so that μ may be less than unity). The relation that obviously must hold between the outputs of one period and the inputs to the next is given by the inequality

$$(1) \qquad Bx \leq A(\mu x)$$

indicating that all inputs must have been produced before using them (we are speaking of a closed economy) though there may be possibly some waste.

In a competitive market there will exist some prices, given by the probability vector $\underline{p}$ (only relative prices interest us, so that normalization of the price system will not affect our results), and some interest rate r, such that no activity pricing

out at a loss will be operated; furthermore, those activities which are used will break even in an equilibrium situation, since otherwise, new firms would enter or leave the market. Thus we have the condition

$$(2) \qquad pB \geq (1 + r)\, pA$$

which states that profits should be non-positive.

We need two further conditions. The first one, that over-produced goods are free, can be writeen, taking (1) into account, as

$$(3) \qquad pBx = p\, A\, (\mu\, x).$$

The second has been already stated: no activity will be operated at a loss; together with (2) we may express this in the equation

$$(4) \qquad pBx = (1 + r)\, pAx$$

It has been Von Neumann's achievement to show for the first time that, under suitable conditions, a simultaneous solution to equations (1) - (4) exists, with the additional result that

$$(5) \qquad \mu = 1 + r$$

He has also shown that μ is the maximal attainable proportional growth factor, and at the same time r is the minimal interest rate which sustains profitless production.

In what follows we shall need the following assumptions, which are usual in the modern treatment of the subject, about the input and output matrices[2]

$$
\begin{aligned}
&\text{a)} \quad A \geq 0, \quad B \geq 0 \\
(6) \qquad &\text{b)} \quad x \geq 0, \quad Ax = 0 \qquad \text{implies } x = 0 \\
&\text{c)} \quad p \geq 0, \quad pB = 0 \qquad \text{implies } p = 0
\end{aligned}
$$

Their interpretation in economic terms is immediate. The first condition means that there are no negative quantities, either of inputs or of outputs, for any of the activities. Condition b) states that, whenever there is no commodity used as input, no activity can be operated at a positive level; in other words, it is impossible to engage in a productive process without using some amount of the existing stocks. Condition c) means that the value of the output of all activities can be zero only when all commodities are free; this is the same as saying that any good can be produced by some process, leaving out the case of non-reproducible commodities

[2] Although these assumptions are sufficient to guarantee the existence of a maximal growth factor, further restrictions are needed to insure equality (5).

because of which the economy, if it needs them as inputs, would run down in one period. Note that storage activities can be considered as productive processes, so that we are leaving out only those non-reproducible resources which when not used up in one period will be wasted.

Our problem becomes that of finding probability vectors $\underline{x}$ and $\underline{p}$ and a positive number μ satisfying relations (1) - (4). Because of the maximality of the growth factor μ, we may try to compute it by solving the problem

$$\begin{aligned} &\max \mu, \text{ subject to} \\ (7) \qquad &(\mu A - B)\, x \leq 0 \\ &-e\, x < -1 \\ &x \geq 0 \end{aligned}$$

where $\underline{e}$ is a vector (sum vector) whose coordinates are all equal to unity. The first restriction corresponds to inequality (1), whereas the others exclude the cases of no production and negative activity levels.

This problem reminds us of a linear program, except for the variable μ which enters in a non-linear way. Thus an algorithm for linear programming such as the simplex procedure cannot be applied directly. Nevertheless, we shall solve a linearized version of problem (7) in order to improve any pair of estimates giving a lower and upper bound for μ, and so that these estimates converge

rapidly to the common solution. At the same time, we will obtain a proof of the existence of a solution.

The first step is to find an interval containing the optimum growth factor μ^0. Any initial bounds could do; a simple way of obtaining them is to consider any pair of positive vectors $\underline{x}$ and $\underline{p}$, for example the sum vector $\underline{e}$, and defining the lower and upper bounds such that (1) and (2) are satisfied. In other words, for the lower bound we have

$$(8) \qquad s_1 = \min_i \left\{ \frac{\sum_j b_{ij}}{\sum_j a_{ij}} \qquad \sum_j a_{ij} > 0 \right\}$$

so that

$$s_1 \, Ae \leq Be$$

satisfies (1), and of course

$$0 < s_1 \leq \mu^0 .$$

The existence of s_1 is guaranteed by assumption (6b), which implies that the matrix $\underline{A}$ is not zero, so that some of its row sums is positive.

For the upper bound we may take

$$(9) \qquad u_1 = \max_j \left\{ \frac{\sum_i b_{ij}}{\sum_i a_{ij}} \right\}$$

so that

$$u_1 \, eA \geq eB,$$

which satisfies (2), with

$$u_1 \geq \mu^0 .$$

Again by assumption (6b), the column sums of $\underline{A}$ must all be positive; hence u_1 is well defined.

These are then the two bounds on μ^0 needed to start the algorithm. We shall proceed by describing the general step in the procedure, letting the index $\underline{t}$ indicate the iteration, so that s_1 and u_1 give the bounds for the first trial, with $\underline{t} = 1$.

Given the bounds s_t and u_t for the t^{th} iteration, compute the new trial value for μ by taking their average

$$u_t = \frac{}{2} (s_t + u_t) . \tag{10}$$

Solve the auxiliary linear programming problem

maximize λ subject to

$$\begin{aligned} (u_t A - B)\, x + \lambda\, e &\leq 0 \\ e\, x &= 1 \\ x &> 0 , \end{aligned} \tag{11}$$

where u_t is now given by (10), and its dual

minimize ρ subject to

$$(12) \qquad p\,(\mu_t A - B) + \rho\, e \leq 0$$

$$p\, e = 1$$

$$p \geq 0 \ .$$

By the duality theorem of linear programming, since both problems are feasible both have and optimal solution, $(x_t\ ,\ \lambda_t)$ and $(p_t\ ,\ \rho_t)$, with the same value for the objective function. That is, $\lambda_t = \rho_t$.

We have to consider two distinct cases

a. If $\lambda_t \geq 0$, we see that

$$(\mu_t A - B)\, x_t < 0$$

so that a new lower bound can be found for μ^o by increasing μ_t . Thus we may take for the next lower bound

$$(13) \qquad s_{t+1} = \min \left\{ \frac{\sum_j b_{ij}\, x_j^t}{\sum_j a_{ij}\, x_j^t} \quad \sum_j a_{ij}\, x_{\cdot}^t > 0 \right\}$$

which satisfies

$$(s_{t+1} A - B)\, x_t \leq 0$$

with an equality for at least one coordinate. Therefore we conclude that

$$s_{t+1} \geq \mu_t > s_t$$

giving us a better estimate for the lower bound. As for the upper bound, set

$$(14) \qquad u_{t+1} = u_t$$

the cycle can now be repeated, with improved bounds on μ^0.

b. If $\lambda_t < 0$, we use the dual restrictions in order to conclude that

$$(15) \qquad p_t \, (u_t A - B) > 0 \, .$$

Hence the new upper bound will be less than the trial value μ_t, and can be taken as

$$(16) \qquad u_{t+1} = \max_j \left\{ \frac{\sum_i p_i^t \, b_{ij}}{\sum_i p_i^t \, a_{ij}} \right\}$$

which will be well defined since (15) implies that the denominator of the expression in brackets is positive for all j.

Furthermore

$$p_t \, (u_{t+1} A - B) \geq 0 \, ,$$

with an equality for at one least coordinate. Consequently

$$u_{t+1} < u_t < u_t$$

and the upper bound has been improved. Taking the same lower bound for the next iteration

(17) $$s_{t+1} = s_t$$

will leave us ready for a new step.

It is evident that at each step the interval between the two bounds shrinks to less than one half the previous length.[3] Since

$$|\mu_1 - u^0| \leq \frac{1}{2}(u_1 - s_1)$$

we must have

$$|u_t - u^0| < 2^{-t}(u_1 - s_1)$$

so that three iterations increase the accuracy by almost one decimal place. In practice the convergence will be considerably faster, especially for the first iterations.

It is of course not necessary to solve each linear programming problem starting at the origin. In the first place, the dual problem (12) will be automatically solved by the simplex method as (11) is solved. Second, each iteration can be started with the basis at which the previous one was terminated; even though the problem will in general not be primal nor dual feasible,

[3] Compare this with reference [1], where the internal shrinks to exactly to one half the previous length, so that convergence will be slower.

a few pivot steps of the dual simplex method followed by the primal simplex method will solve it. The computational effort will thus not be too great, except for the first few iterations. The source of the largest amount of computations will be the reinversion of the basis, since at each step there will be a change in the coefficients of the matrix.

It is evident that the set of solutions of (1) with s_t substituted for μ converges to the Von Neumann facet; at each step s_t is either increased or left constant, so that the set of activity levels satisfying these inequalities will shrink, each containing its predecessor. Since each set is closed and their intersection is non-empty, the limiting set exists, and satisfies the same inequality with $\mu = \mu^0$. The dual argument shows that the solutions of (2) converge to the Von Neumann prices.

It is very likely that the convergence of the algorithm could be improved by trying to predict the outcome of several steps. This will be so particularly for the later iterations. The following linear extrapolation could be tried from time to time, whenever the trial value μ_t underestimates the new lower bound s_{t+1} for some iterations prior to $t = T$.

Assume

$$(18) \qquad s_t = \alpha + \beta \mu_t$$

for $t \geq T$, and assume also that u_T , s_T , s_{T+1} , u_T are known.

Then, using (18) for $t = T$ and $t = T + 1$, we find

$$\alpha = \frac{s_T \mu_{T+1} - s_{T+1} u_T}{\mu_{T+1} - u_T}$$

$$\beta = \frac{s_{T+1} - s_T}{u_{T+1} - \mu_T}$$

On the other hand, by definition, for $t > T$

$$u_{t+1} = \frac{1}{t+1} (s_t + \mu_t)$$

$$= \frac{1}{t+1} [\alpha + u_T + \beta u_t]$$

or, if $\beta < 2$, letting t tend to infinity

$$u_\infty = \frac{1}{2} [\alpha + u_T + \beta \mu_\infty] = \frac{\alpha + u_T}{2 - \beta}$$

Thus we may set

$$u_{T+2} = u_\infty$$

for accelerated convergence.

A similar device would accelerate convergence when the

lower bound remained unchanged for several iterations. It would also be possible to use more refined extrapolations.

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