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**Fecha de publicación**: 2023
**Publicado como artículo en:** Discrete Applied Mathematics 339. 107-126 //ISSN-e: 1872-6771

# A branch-and-cut algorithm for the routing and spectrum allocation problem

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**Abstract**

One of the most promising solutions to deal with huge data traffic demands in large communication networks is given by flexible optical networking, in particular the *flexible grid* (flexgrid) technology specified in the ITU-T standard G.694.1. In this specification, the frequency spectrum of an optical fiber link is divided into narrow frequency *slots*. Any sequence of consecutive slots can be used as a simple channel, and such a channel can be switched in the network nodes to create a *lightpath*. In this kind of networks, the problem of establishing lightpaths for a set of end-to-end demands that compete for spectrum resources is called the *routing and spectrum allocation problem* (RSA). Due to its relevance, RSA has been intensively studied in the last years. It has been shown to be NP-hard and different solution approaches have been proposed for this problem. In this paper we present several families of valid inequalities, valid equations, and optimality cuts for a natural integer programming formulation of RSA and, based on these results, we develop a branch-and-cut algorithm for this problem. Our computational experiments suggest that such an approach is effective at tackling this problem.

*Keywords:* integer programming, valid inequalities, networking, optical fibers, flex-grid
*2000 MSC:* 90C10, 94A05

## 1. Introduction

Elastic optical networks seem to be indispensable to lead with the ever-increasing bandwidth demands due to its many desirable properties including flexible data rate and spectrum allocation, low signal attenuation, low signal distortion, low power requirement, low material usage, small space requirement, and low cost [1]. Such a network translates the known routing and wavelength

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allocation problem into the *routing and spectrum allocation* (RSA) [2, 3] which in its simplest version consists in establishing the lightpaths for a set of end-to-end traffic demands that, assuming the same modulation format for each, are expressed in terms of the number of required slots. Since lightpaths are determined by a route and a selected channel that satisfies the volume, RSA involves finding a route and assigning frequency slots to each demand. Several solution approaches have been explored in the last years, including direct approaches using commercial ILP-solvers to solve formulations to optimality [2, 4, 5, 6, 7], or to solve ILP based heuristics [8, 9, 10, 11, 12], generally using pre-computed sets of paths. Several heuristics [13, 14, 15, 16, 17, 18, 19] and metaheuristics [20, 21, 22] like genetic algorithms [12, 23] have also been proposed. A few works can be found where the authors apply Lagrangian decompositions and column generation techniques [14, 24, 25, 26, 27, 28, 29, 30].

The most common approach is to decompose the RSA problem into two phases or sub-problems. Since this approach tries to reduce the search space, the first phase is generally solved by calculating the set of $k$-shortest paths for each demand. Once a path for each demand is determined, the problem that remains to be solved is the *spectrum allocation* problem (SA) [31, 32, 33], which consists in assigning the spectrum to each demand, having the route already fixed. This problem belongs to the class $\mathcal{NP}$-hard even for paths, if their length is of at least three arcs, and for any kind of rings [13, 34]. However, each of these two sub-problems or phases is probably easier to solve in practice than the original RSA and can provide some bounds to the optimal value.

Formally, we are given a digraph $G = (V, E)$ representing the optical fiber network, a fixed number $\bar{s} \in \mathbb{Z}_+$ of available slots, and a set of *demands* $D = \{d_i = (s_i, t_i, v_i)\}_{i=1}^k$, where each demand $d_i$, $i = 1, \ldots, k$, is composed by a source $s_i \in V$, a target $t_i \in V$, and a volume $v_i \in \mathbb{Z}_+$. We define a *lightpath* for a demand $d_i = (s_i, t_i, v_i)$ to be a tuple $(l, r, p)$, where $1 \leq l \leq l + v_i - 1 \leq r \leq \bar{s}$ and $p$ is a (directed) path in $G$ from $s_i$ to $t_i$. We say that two lightpaths $(l, r, p)$ and $(l', r', p')$ overlap if $p \cap p' \neq \emptyset$, $l' < r$, and $l < r'$. In this setting, RSA consists in establishing a lightpath associated to each demand, in such a way that lightpaths do not overlap.

### *1.1. Integer programming formulation*

To the best of our knowledge, most of the works carried out are heuristics, and although several authors presented various mathematical models to solve the RSA -or its variants- to optimality [12, 14, 35, 36, 37, 38], they have hardly taken advantage of the great power that integer linear programming techniques provide. In fact, we have found very few works that present valid inequalities to improve the generic branch-and-cut implementation of ILP solvers [28, 39, 40, 41, 42, 43].

In this work, we concentrate on the so-called *demand-slot-link basic formulation* (DSL-BF), which was one of the best-performing models in the experiments reported in [44]. The family of binary variables $u \in \{0, 1\}^{|D||E|\bar{s}}$ is defined for every demand $d \in D$, every arc $e \in E$, and every slot $s \in S$, in such a way that $u_{des} = 1$ if and only if the demand $d$ uses the slot $s$ over the arc $e$. For technical reasons, we also consider the fictitious variable $u_{de,\bar{s}+1}$ that always takes value 0, for every $d \in D$ and every $e \in E$.

Likewise, when dealing with the static RSA problem, several different objective functions may be of interest for the network operator. In the present work we seek to minimize the length of the routes assigned to each demand, as in works [10, 42, 45, 46], among others.

If $d = d_i = (s_i, t_i, v_i) \in D$ is a demand, for some $i \in \{1, \ldots, k\}$, we define $s(d) = s_i$, $t(d) = t_i$, and $v(d) = v_i$. For $j \in V$, we define $\delta^-(j)$ to be the set of incoming arcs to $j$, and $\delta^+(j)$ to be the set of outgoing arcs from $j$. In this setting, the DSL-BF formulation for RSA is the following integer program.

$$\min \quad \sum_{d\in D}\sum_{e\in E}\sum_{s\in S} u_{des}/v(d) \tag{1}$$

$$\text{s.t.} \quad \sum_{e\in\delta^-(j)} u_{des} - \sum_{e\in\delta^+(j)} u_{des} = 0 \qquad \forall s \in S, \forall d \in D, \forall j \in V \setminus \{s(d), t(d)\} \tag{2}$$

$$\sum_{e\in\delta^+(s(d))}\sum_{s\in S} u_{des} \geq v(d) \qquad \forall d \in D \tag{3}$$

$$\sum_{e\in\delta^-(s(d))}\sum_{s\in S} u_{des} = 0 \qquad \forall d \in D \tag{4}$$

$$\sum_{d\in D} u_{des} \leq 1 \qquad \forall e \in E, \forall s \in S \tag{5}$$

$$u_{de,\bar{s}+1} = 0 \qquad \forall d \in D, \forall e \in E \tag{6}$$

$$v(d)(u_{des} - u_{de,s+1}) \leq \sum_{s'=f}^{s} u_{des'} \qquad \forall d \in D, \forall e \in E, \forall s \in S, f = \max\{1, s - v(d) + 1\} \tag{7}$$

$$u_{des} \in \{0, 1\} \qquad \forall d \in D, \forall e \in E, \forall s \in S \tag{8}$$

The objective function (1) asks for the total length of the lightpaths to be minimized. Since a solution using spurious cycles uses more arcs than the strictly needed, this objective function also forbids spurious cycles in any optimal solution. Constraints (2) impose flow conservation restrictions for each demand at each node in the network, except for the source and sink nodes associated with the demand. Constraints (3) ensure that the required number of slots is routed for each demand, whereas constraints (4) forbid incoming arcs into the source node of each demand with the aim of forcing each path to end in the destination node. Otherwise a cycle containing the source would satisfy all the constraints. Constraints (5) guarantee that no two different lightpaths overlap. Finally, constraints (6) and (7) ensure slot contiguity in the following way. If a demand has assigned a slot $s$ in an arc but not the slot $s + 1$, then it must have assigned the previous $v(d)$ slots. Constraints (6) allows us to also apply the inequalities to the last available slot $\bar{s}$.

We define $RSA(G, D, \bar{s})$ to be the convex hull of feasible solutions of the DSL-BF formulation (2)-(8). A *valid inequality* (resp. *valid equation*) is a linear inequality (resp. equation) on $u$ satisfied by all points in $RSA(G, D, \bar{s})$. An *op-*

*timality cut* (resp. *optimality equation*) is a non-valid inequality (resp. equation) that does not remove all optimal solutions.

The polytope $RSA(G, D, \bar{s})$ seems hard to study, since even characterizing its dimension is not straightforward (as suggested by the valid equations in the next sections). Due to this fact, in the present work we explore families of valid inequalities without providing facetness results. Instead, we aim for the more modest goal of showing that –together with the model constraints– the inequalities from these families do not imply each other. We have also implemented a branch-and-cut procedure for RSA in order to evaluate the contribution of the proposed families of valid inequalities, valid equations, and optimality cuts within a cutting plane environment.

The remainder of this work is organized as follows. Section 2 presents some theoretical results, Sections 3, 4, and 5 introduce several families of valid inequalities and equalities, and optimality cuts based on flow, contiguity, and non-overlapping considerations, respectively. Section 6 presents the branch-and-cut algorithm. Section 7 summarizes the results of the computational experiments performed in order to measure the efficiency of adding each family of cuts. Section 8 presents some conclusions and possible future work.

## 2. Properties coming from symmetry considerations

The DSL-BF formulation presents several symmetries; for example, for any given solution, flipping the spectrum upside-down would generate a solution of same cost. These consideration lead to the following technical results. Let $u' \in \mathbb{R}^{|D||E|\bar{s}}$, we define as $I_s : \mathbb{R}^{|D||E|\bar{s}} \to \mathbb{R}^{|D||E|\bar{s}}$ the function such that if $\bar{u} = I_s(u')$ then $\bar{u}_{des} = u'_{de,\bar{s}-s+1}$ for every arc $e \in E$, demand $d \in D$, and slot $s \in S$.

**Theorem 2.1.** *If $a \cdot u \leq b$ is a valid inequality (resp., optimality cut) for $RSA(G, D, \bar{s})$, then the* slot-symmetrical inequality

$$\sum_{d \in D} \sum_{e \in E} \sum_{s \in S} a_{des} \; u_{de,\bar{s}-s+1} \leq b \tag{9}$$

*is also valid (resp., an optimality cut) for $RSA(G, D, \bar{s})$. Furthermore, if $a \cdot u \leq b$ induces a facet of $RSA(G, D, \bar{s})$, then (9) also induces a facet of this polytope.*

*Proof.* If $u'$ is a feasible solution for $RSA(G, D, \bar{s})$ its transformation $I_s(u')$ is also feasible and has the same objective value, thus if $a \cdot u \leq b$ is a valid inequality (resp. optimality cut), since (9) is exactly $a \cdot I_s(u) \leq b$, then (9) is also a valid inequality (resp. optimality cut). Assuming that the dimension of $RSA(G, D, \bar{s})$ is $n$, if $a \cdot u \leq b$ is facet-defining, there must exist $n$ affinely independent vectors that satisfy $a \cdot u_i = b$, with $i = 1, \ldots, n$, so since the function $I_s$ preserves the affinity of the given vector set, their transformations $I_s(u_1), \ldots, I_s(u_n)$ are also affinely independent and satisfy $a \cdot I_s(u_i) = b$ for $i = 1, \ldots, n$, and therefore, the inequality (9) also induces a facet of the polytope. $\square$

**Theorem 2.2.** *Fix $d \in D$ and define $\delta_d = \delta(t(d)) \cup \delta(s(d))$. Let $a \cdot u \leq b$ be an optimality cut for $RSA(G, D, \bar{s})$ such that for every $s \in S$ there exist $\alpha_{ds} \in \mathbb{R}$ and $\beta_{ds} \in \mathbb{R}$, with $a_{des} = \alpha_{ds}$ for every $e \in \delta^+(s(d))$, $a_{des} = \beta_{ds}$ for every $e \in \delta^-(t(d))$, $a_{des} = 0$ for every $e \in \delta^-(s(d)) \cup \delta^+(t(d))$, and $a_{d'es} = 0$ for every $d' \neq d$ and $e \in \delta_d$. Then, the inequality*

$$\sum_{s \in S} \Big( \sum_{e \in \delta^-(t(d))} \alpha_{ds} u_{des} + \sum_{e \in \delta^+(s(d))} \beta_{ds} u_{des} + \sum_{d' \in D} \sum_{e \in E \setminus \delta_d} a_{d'es} u_{d'es} \Big) \leq b \quad (10)$$

*is also an optimality cut for $RSA(G, D, \bar{s})$.*

*Proof.* It is enough to prove that in every optimal solution $u^*$, the equality

$$\sum_{e \in \delta^-(t(d))} u^*_{des} = \sum_{e \in \delta^+(s(d))} u^*_{des}. \quad (11)$$

is satisfied for every $d \in D$ and $s \in S$. Due to the flow conservation constraints (2) we can verify that if $u^*_{des} = 1$ with $e \in \delta^+(s(d))$, then there must exist a path $P \subseteq E$ with $e \in P$ such that $u^*_{de's} = 1$ for every $e' \in P$. Because of the constraints (3), the node $s(d)$ cannot belong to that path, so it must inevitably end in $t(d)$. Also, since no optimal solution can contain a cycle, in which case it would not be optimal, then $\sum_{e \in \delta^+(t(d))} u^*_{des} = 0$. Therefore, if all paths on the slot $s$ that start in $s(d)$ finish in $t(d)$ and since no cycle can exist beginning and ending in $t(d)$, the inequality (11) is satisfied, and thus it is possible to transform $a \cdot u \leq b$ into (10). ☐

## 3. Flow inequalities and optimality cuts

In this and the following sections we propose several families of valid inequalities, valid equations, and optimality cuts for $RSA(G, D, \bar{s})$. The formulation does not include constraints avoiding spurious elements, as cycles, bifurcations, more than one lightpath and more than $v(d)$ slots satisfying a demand $d$, for $d \in D$. Instead, we rely on the objective function for this task, as no optimal solution will contain such elements. Due to this fact, many of the inequalities presented in this work are not valid but are optimality cuts instead, since they apply to solutions with no spurious elements and exactly $v(d)$ slots in the only lightpath satisfying each demand $d$, for $d \in D$. We might add to the model suitable constraints avoiding spurious elements, and in this case all the optimality cuts in this work become valid inequalities for such a strengthened model.

Because of the theorems enunciated in Section 2, each time we have a family of inequalities related to the outgoing arcs of a particular node, we can formulate the symmetrical one looking at the ingoing arcs to that node. Likewise, when we state a family of inequalities that include in any way the lowest usable slot, we can state a symmetrical inequality by considering the highest usable slot.

Whenever we have a particular vector $u' \in \{0,1\}^{|D||E|\bar{s}}$, we write $u'_{de*} \in \{0,1\}^{\bar{s}}$ to be the sub-vector of $u'$ that refers to the slots used by the demand $d$ in the arc $e$. Analogously, $u'_{d**} \in \{0,1\}^{|E|\bar{s}}$ is the sub-vector of $u'$ that refers to all the slots and arcs used by $d$. We denote as $|C|$ the number of arcs of $C$ for every set $C \subseteq E$. A *simple path* is a path without cycles; we call as $P(G, i, j)$ the set of simple paths from $i$ to $j$ in $G$, and $E(G, d)$ the set composed by every arc $e$ such that there exists a path $P \in P(G, s(d), t(d))$ and $e \in P$. Due to the space restrictions, we omit most of the proofs. None of the presented families is implied by the model constraints. Despite a large number of the families presented are not mutually implicated –which was specifically stated for those we consider less clear–, we also included sub-families or families clearly dominated by others because this allows us to have a finer adjustment in the separation procedures. Even more, some of these dominated families obtained better results than their dominant ones.

**Theorem 3.1.** *The equalities* flow I*, namely,*

$$\sum_{s \in S} \sum_{e \in \delta^+(t(d))} u_{des} = 0 \qquad \forall d \in D, \tag{12}$$

*do not cut any optimal solution from $RSA(G, D, \bar{s})$.*

*Proof.* Let $u^*$ be an optimal solution, and assume that for some $d \in D$ there exists an arc $e \in \delta^+(t(d))$ such that $\sum_{s \in S} u^*_{des} > 0$. As $u^*$ is optimal, the demand $d$ is satisfied by a lightpath composed by a simple path $P$ from $s(d)$ to $t(d)$ with at least $v(d)$ consecutive slots on (i.e., slots such that their corresponding variables equal 1). Since $e \notin P$, then it is possible to define a new feasible solution $u'$ such that $u'_{de's} = u^*_{de's}$ for every $s \in S$ and for every $e' \in P$, $u'_{de's} = 0$ for every $e' \notin P$ and for every $s \in S$, and $u'_{d'**} = u^*_{d'**}$ for every $d' \neq d$. Since $d$ does not use $e$ in $u'$, then the objective value of $u'$ is strictly better than the objective value of $u^*$, a contradiction. □

**Theorem 3.2.** *The family of inequalities* flow II*, namely,*

$$\sum_{e \in \delta^+(i)} u_{des} \leq 1 \qquad \forall d \in D,\ \forall s \in S,\ \forall i \in V \setminus \{t(d)\}, \tag{13}$$

*do not remove any optimal solution from $RSA(G, D, \bar{s})$.*

*Proof.* Let $u^*$ be an optimal solution such that $u^*$ violates at least one of the inequalities (13). Then there must exist at least one node $i \in V$ such that the sum of the variables related to a slot $s$ over $\delta^+(i)$ is greater than 1. That means that exists a bifurcation, therefore two paths. Because of the model constraints (4) and the flow conservation constraints (2), each of them must have at least $v(d)$ slots used, and as well as they share slots, they must be disjoint in arcs. Thus, we can define a new solution $u'$ that uses only one of these paths, i.e., setting 0 to every slot of the arcs belonging to the other path, with objective function strictly lower than $u^*$, a contradiction. □

**Corollary 3.1.** *We call* flow III *to the sub-set of optimality cuts obtained by taking $i = s(d)$, in (13), namely,*

$$\sum_{e \in \delta^+(s(d))} u_{des} \leq 1 \qquad \forall d \in D,\ \forall s \in S. \tag{14}$$

**Corollary 3.2.** *Due to the flow conservation constraints (2) and Theorem 2.2, the symmetrical inequality obtained by taking the sum on every* $e \in \delta^-(i), i \in V \setminus \{s(d)\}$*, called* flow IV*, and the case when* $i = t(d)$*, i.e.,* flow V*, are also optimality cuts for the model DSL-BF.*

**Definition 3.1.** *Let* $u'$ *be a solution for the RSA. A lightpath* $(P, C)$ *for a demand* $d$ *is said to be minimal if* $P$ *is a simple path from* $s(d)$ *to* $t(d)$*,* $C$ *is a channel of exactly* $v(d)$ *slots, and there does not exist another solution* $u''$ *using the same lightpaths as* $u'$ *for every other demand different than* $d$*, but using a lightpath* $(P', C)$ *to satisfy* $d$ *where* $P'$ *is composed of fewer arcs than* $P$*.*

**Lemma 3.1.** *Any feasible solution such that there exists a demand satisfied with a non-minimal lightpath is not optimal.*

**Theorem 3.3.** *The inequalities* flow-volume VI*, namely,*

$$\sum_{e\in\delta^+(i)} \sum_{s\in S} u_{des} \leq v(d) \qquad \forall d \in D,\ \forall i \in V, \tag{15}$$

*do not remove any optimal solution from* $RSA(G,\ D, \bar{s})$*.*

*Proof.* Let $u^*$ be an optimal solution removed by at least one of the inequalities (15). Because of its optimality and due to Lemma 3.1, every demand $d$ must be satisfied by a minimal lightpath, but the inequalities violation claims that the number of slots used by some demand $d'$ in $\delta^+(i')$ for some $i' \in V$ must be at least $v(d') + 1$, a contradiction. □

**Corollary 3.3.** *We call* flow-volume VII *to the sub-set of optimality cuts obtained by taking* $i = s(d)$ *in (15); namely,*

$$\sum_{e\in\delta^+(s(d))} \sum_{s\in S} u_{des} \leq v(d) \qquad \forall d \in D, \tag{16}$$

**Corollary 3.4.** *Due to Theorem 2.2, we can also state that exactly* $v(d)$ *slots must reach the destination of every demand* $d \in D$*, by taking* $e \in \delta^-(t(d))$ *on (16). We can also state the same for the incoming arcs of every node in the graph, i.e., the analogous to (15). We name these families as* flow-volume VIII *and* flowflow-volume IX*, respectively, and they are also optimality cuts.*

**Theorem 3.4.** *The inequalities* flow-branches X*, namely,*

$$\sum_{e'\in\delta^+(i)\setminus\{e\}} \sum_{s'\in S} u_{de's'} \leq v(d)(1 - u_{des}) \qquad \begin{array}{c}\forall d \in D,\ \forall i \in V, \\ \forall e \in \delta^+(i),\ \forall s \in S,\end{array} \tag{17}$$

*are optimality cuts for the model* DSL-BF*.*

*Proof.* Let $u^*$ be an optimal solution. If $u^*$ violates at least one of the inequalities (17), then, there must exist a demand $d \in D$ and a node $i \in V$ such that $d$ uses at least two arcs of $\delta^+(i)$. Due to the contiguity and integrality constraints both arcs must have at least $v(d)$ slots used by $d$, and due to flow limitations, each must belong to either a cycle or a path that connects $s(d)$ with $t(d)$. Since a demand in any optimal solution must be satisfied by a lightpath composed by a simple path and a channel with exaclty $v(d)$ slots, then $u^*$ cannot be optimal. □

**Corollary 3.5.** *We call* flow-branches XI *to the inequalities obtained by taking* $i = s(d)$ *in (17), namely,*

$$\sum_{e' \in \delta^+(s(d)) \setminus \{e\}} \sum_{s' \in S} u_{de's'} \leq v(d)(1 - u_{des}) \qquad \begin{array}{r} \forall d \in D,\, \forall e \in \delta^+(s(d)), \\ \forall s \in S. \end{array} \tag{18}$$

Assume that the constraints (2)-(7) hold, then the family *flow-branches XI* does not imply *flow-volume VII*, nor vice versa. Likewise, if the constraints (2)-(7) hold, then, the inequalities (17) do not imply (15), nor vice versa.

**Corollary 3.6.** *Because of the flow conservation constraints (2) and Theorem 2.2, we can look at the incomming arcs,* $\delta^-(i)$ *on every* $i \in V \setminus s(d)$*, obtaining the family called* flow-branches XII*. We call* flow-branches XIII *the sub-family obtained when* $i = t(d)$*. Both families are optimality cuts for the model DSL-BF.*

**Theorem 3.5.** *The inequalities* flow-used-arcs XIV*, namely,*

$$\sum_{e \in E} u_{des'} \leq \sum_{e \in E} u_{des} + |E|\Big(1 - \sum_{e \in \delta^+(s(d))} u_{des}\Big) \qquad \begin{array}{r} \forall s, s' \in S,\, s' \neq s, \\ \forall d \in D, \end{array} \tag{19}$$

*are optimality cuts for the model DSL-BF.*

*Proof.* Let $u^*$ be an optimal solution that violates at least one of the inequalities (19) for the demand $d$ and the slot $s$. Since the integer vector $u^*$ satisfies the inequalities (13) due to Theorem 3.2, then $\sum_{e \in \delta^+(s(d))} u_{des} \in \{0, 1\}$. In order to violate (19), if this sum is 0, the amount of arcs in which $s'$ is used must be greater than the total number of arcs of the graph; therefore $\sum_{e \in \delta^+(s(d))} u_{des} = 1$. This implies that the slot $s$ is used by $d$ and that there exists $s' \neq s$ which is used by $d$ in more arcs than $s$ in $u^*$. As $u^*$ is an optimal solution, due to Lemma 3.1 the demand $d$ must be satisfied by a minimal lightpath composed by a path $P$ and a channel of $v(d)$ slots, and thus $s$ and $s'$ have to be used only in $P$ so in the same amount of arcs, a contradiction. □

**Theorem 3.6.** *Inequalities* flow-used-arcs XV*, namely,*

$$\frac{1}{v(d)}\Big(\sum_{e \in E} \sum_{s' \in S} u_{des'}\Big) \leq \sum_{e \in E} u_{des} + |E|\Big(1 - \sum_{e \in \delta^+(s(d))} u_{des}\Big) \qquad \begin{array}{r} \forall d \in D, \\ \forall s \in S, \end{array} \tag{20}$$

*are optimality cuts for the model DSL-BF.*

*Proof.* Let $u^*$ be an optimal solution and assume it violates some of the inequalities (20). Since vector $u^*$ satisfies the inequalities (13) due to Theorem 3.2 and $u^* \in \{0,1\}^{|D||E|\bar{s}}$, therefore the sum $\sum_{e\in\delta^+(s(d))} u^*_{des} = \alpha$ must be 0 or 1 for every $s \in S$ and $d \in D$. If $\alpha = 0$, then there must exist a demand $d \in D$ and a slot $s \in S$ such that $\sum_{e\in E}\sum_{s'\in S} u^*_{des'} > v(d)|E|$, so, $u^*$ is not optimal because it uses more slots than $v(d)|E|$. If $\alpha = 1$, then there must exist a demand $d \in D$ and a slot $s \in S$ such that

$$\sum_{e\in E}\sum_{s'\in S} u^*_{des'} > v(d) \sum_{e\in E} u^*_{des}, \tag{21}$$

but $\sum_{e\in E} u^*_{des} \geq 1$, because slot $s$ is used by $d$ in $\delta^+(s(d))$. Since $u^*$ is optimal, due to Lemma 3.1, $d$ must be satisfied by a minimal lightpath composed by a channel $C$ and a path $P$ that connects $s(d)$ with $t(d)$. Since the slot $s$ must belong to $C$, it must be used all along the path $P$, so the right-hand side of (21) is at least $v(d)|P|$. But the number of slots of the lightpath composed by $P$ and $C$ is enough to satisfy $d$, and it is exactly $v(d)|P|$, therefore (21) cannot be satisfied for any optimal solution, then $u^*$ is not optimal, a contradiction. □

Assume that the constraints (2)-(7) and the inequalities (13) and (15) hold. Then, the inequalities (20) are not implied by (19). Let $u'$ be a solution of an instance of RSA with only one demand $d$ with $v(d) = 1$ that is satisfied using a path of two arcs that connects $s(d)$ with $t(d)$ and using slot 1, and a spurious cycle of two arcs $ij$ and $ji$ using slot 2. Vector $u'$ is a feasible solution that satisfies the inequalities (13) and (15) because for each vertex $k \in V$, $d$ uses at most one arc of $\delta^+(k)$ with at most $v(d)$ slots. Since each slot is used in exactly two arcs, $u'$ also satisfies (19). However, $u'$ does not satisfy (20) because it uses 4 slots in all the graph, which is greater than $v(d)$ times the amount used by each slot, i.e., 2.

Note that we could strengthen the inequalities (19) and (20) by considering $|E(G,d)|$ instead of $|E|$, but since the graphs we used in the experiments are strongly connected, both sets are mostly the same.

Given an arc sub-set $E' \subseteq E$, we call $P_{E'}$ to the set of all paths in $E'$ without cycles, and we call $M(P_{E'}) \subseteq \mathcal{P}(E')$ to a set of vertex-disjoint paths with maximum total number of arcs. If $|M(P_{E'})|$ is the total amount of arcs of the set $M(P_{E'})$, then each demand in any optimal solution can use at most $|M(P_{E'})|$ arcs on $E'$. This results in the large family of inequalities shown in (22). By also forcing each demand to be satisfied by exactly its volume, we get the optimality cuts shown in (23). A particular case of this family is obtained by taking $E' = \delta^+(i)$ for every $i \in V$, i.e., (15).

$$\sum_{e\in E'} u_{des} \leq |M(P_{E'})| \qquad \forall d \in D, \forall s \in S, \forall E' \subseteq E, \tag{22}$$

$$\sum_{s\in S}\sum_{e\in E'} u_{des} \leq v(d)|M(P_{E'})| \qquad \forall d \in D, \forall E' \subseteq E. \tag{23}$$

**Theorem 3.7.** *The inequalities (22) are optimality cuts for the model DSL-BF.*

*Proof.* Let $u^*$ be an optimal solution that violates at least one of the inequalities of this family. So, there must be a demand $d \in D$, a slot $s \in S$ and a set of arcs $E' \subseteq E$ such that $\sum_{e \in E'} u^*_{des} > |M(P_{E'})|$. That is, the demand $d$ uses the slot $s$ in a set of arcs $Y \subseteq E'$ such that $|Y| > |M(P_{E'})|$. However, since $u^*$ is optimal, its path cannot contain cycles nor bifurcations, so $Y$ must be a set of paths, i.e., $Y \subseteq P_{E'}$, so $|Y| \leq |M(P_{E'})|$. Thus, $u^*$ cannot be optimal. □

**Theorem 3.8.** *The inequalities (23) are optimality cuts for the model DSL-BF.*

*Proof.* Let $u^*$ be an optimal solution for RSA and assume that $u^*$ violates some of the inequalities (23). So, there must exist $d \in D$ and $E' \subseteq E$ such that $\sum_{s \in S} \sum_{e \in E'} u^*_{des} > v(d)\ |M(P_{E'})|$. Let $Y \subseteq E'$ be the set of arcs used by $d$ in $E'$. Since $u^*$ is optimal, due to Lemma 3.1, $d$ must be satisfied by a minimal lightpath with a path $P$ and channel $C$, so $Y \subseteq P$ cannot contain cycles nor bifurcations, i.e., $Y \subseteq P_{E'}$, thus $|Y| \leq |M(P_{E'})|$. It is also true that the channel used by $d$ must be $C$ all along the path $P$, in particular in $Y$, and since $d$ cannot use any arc $e \notin P$, then $\sum_{s \in S} \sum_{e \in E'} u^*_{des} = v(d)|Y|$, therefore $u^*$ cannot be optimal. □

We can prove that the model given by constraints (2)-(7) and the inequalities (23) does not imply the family (22).

Given an arc $ij \in E$, we define $\delta^-(i) \cup \delta(j)$ to be the *incoming double broom* (InDBroom) associated with $ij$. Similarly, we define $\delta^+(i) \cup \delta(j)$ to be the *outgoing double broom* (OutDBroom) associated with $ji$. Note that the arc $ij$ must exist in order to have an InDBroom. Analogous with $ji$ for the OutDBroom.

We can see that if $DB$ is either an InDBroom or an OutDBroom in a graph $G$, then $|M(P_{DB})| = 3$, i.e., the maximum amount of arcs of $DB$ used by any demand in an optimal solution must be lower than or equal to 3.

**Corollary 3.7.** *The particular cases of (23) in which $E'$ is an InDBroom or an OutDBroom are called* flow-dbrooms XVI *and* flow-dbrooms XVII*, respectively.*

Since $M(P_{E'})$ is a set of paths, the total number of arcs belonging to them, i.e., $|M(P_{E'})|$, must be lower than $|V(E')|$, wherewith (24) and (25) are dominated by the inequalities (22) and (23), respectively.

$$\sum_{e \in E'} u_{des} \leq (|V(E')| - 1) \qquad \forall d \in D, \forall s \in S, \forall E' \subseteq E, \tag{24}$$

$$\sum_{s \in S} \sum_{e \in E'} u_{des} \leq v(d)(|V(E')| - 1) \qquad \forall d \in D, \forall E' \subseteq E. \tag{25}$$

**Corollary 3.8.** *In our experiments, we consider the sub-families of (24) and (25) generated by taking $E'$ to be: (I) an undirected cycle in $G$ (i.e., disregarding the arc orientations), getting the sub-families* flow-cycles XVIII *and* flow-cycles XIX *respectively, (II) the set of all edges induced by the vertices of a cycle in $G$, getting the sub-families of inequalities* flow-cycles XX *and* flow-cycles XXI*, respectively, and (III) the set $\{ij, ji\}$, when both arcs exist between the nodes $i$*

*and $j$, obtaining the sub-families* flow-cycles XXII *and* flow-cycles XXIII*, respectively.*

Note that, in spite that (II) can be seen as a lifted version of (I) –and hence (II) clearly dominates (I)–, in the experiments the two families of (I) obtained better results than the two families of (II); and, even more, no family of (I) overcame a generic branch-and-bound, while a family of (I) did it.

**Corollary 3.9.** *Finally, for two arcs $ij, ji \in E$ (when such a structure exists), we consider the family of inequalities* flow-cycles XXIV*, namely,*

$$\sum_{s' \in S} u_{d,ji,s'} \leq v(d)(1 - u_{d,ij,s}) \qquad \forall ij, ji \in E,\ \forall d \in D,\ \forall s \in S. \tag{26}$$

Constraints (2)-(7) and (25) do not imply the family (15), nor vice versa. The inequalities (26) are not implied by the model constraints (2)-(7) together with (23). Since (26) only avoid the cycles including two arcs $ij$ and $ji$ and using more than $v(d)$ slots in arcs with reverse, any solution that uses more than $v(d)$ slots on a cycle with no such a structure, i.e., where no arc of the cycle has a reverse, violates (15) but not (26). Therefore, the model constraints (2)-(7) with the inequalities (26) do not imply (15). Likewise, the model constraints (2)-(7) together with (26) do not imply (23) nor (25).

## 4. Contiguity inequalities

In this section we present valid inequalities and equations dealing with the contiguity requirements, namely that each demand must use consecutive slots. Most of the following families are based on a result shown in [44], called the *consecutive ones* theorem, where given a demand $d$, the contiguity constraint is obtained mainly by grouping the slots that are distanced from each other by the volume of $d$.

**Theorem 4.1.** *The family* contiguity I *and its symmetrical called* contiguity II*, namely,*

$$\sum_{\substack{s \in \{1,\dots,i\}: \\ s \,\equiv\, i \pmod{v(d)}}} u_{des} \geq \sum_{\substack{s \in \{1,\dots,i-1\}: \\ s+1 \,\equiv\, i \pmod{v(d)}}} u_{des} \qquad \begin{array}{r} \forall d \in D, \\ \forall e \in E, \\ \forall i \in S \end{array} \tag{27}$$

$$\sum_{\substack{s \in \{\bar{s}-i+1,\dots,\bar{s}\}: \\ s \,\equiv\, \bar{s}-i+1 \pmod{v(d)}}} u_{des} \geq \sum_{\substack{s \in \{\bar{s}-i+2,\dots,\bar{s}\}: \\ s-1 \,\equiv\, \bar{s}-i+1 \pmod{v(d)}}} u_{des} \qquad \begin{array}{r} \forall d \in D, \\ \forall e \in E, \\ \forall i \in S, \end{array} \tag{28}$$

*respectively, are optimality cuts for the model DSL-BF.*

*Proof.* Let $u^*$ be an optimal solution for an instance of RSA such that $u^*$ violates at least one of the inequalities (27). Then, there must exist a demand $d \in D$, an arc $e \in E$ and a slot $i \in S$ such that the left-hand side is strictly smaller than the right-hand side. If $d$ does not use $e$ in $u^*$, the inequality is satisfied, so $d$ must use $e$, and as it is optimal there must be $v(d)$ contiguous slots used by $d$ in that arc. Since $u^*$ must also satisfy integrality, then $u_{des} \in \{0, 1\}$ for every $s \in S$, and because the slots in the left-hand side are taken distanced by $v(d)$ slots, only one can be used. So the right-hand side must use at least two slots because of the integrality, but the slots in this side are also at a distance of $v(d)$ each other, so only one can be used. Since inequalities (28) are symmetric by slots to (27), thus, due to Theorem 2.1, (28) are also optimality cuts for the model DSL-BF. $\square$

Given the constraints of the model DSL-BF, two counterexamples show that the inequalities (27) do not imply (28) nor vice versa. Consider an instance of the RSA problem with $\bar{s} = 5$, and a graph with only one arc $e$ connecting the unique pair of nodes $i$ and $j$ which are the source and the destination of the unique demand $d$. If $v(d) = 2$, the vector $u'_{de*} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ satisfies the inequalities (27) besides the constraints of the model DSL-BF but not (28). Defining a vector $u''$ by reversing $u'_{de*}$ we have a counterexample for the converse assertion.

**Theorem 4.2.** *The equalities* contiguity III*, which assume that each demand $d \in D$ uses exactly $v(d)$ slots, namely,*

$$\sum_{\substack{s \in S: \\ s \,\equiv\, i \pmod{v(d)}}} u_{des} = \sum_{\substack{s \in S: \\ s+1 \,\equiv\, i \pmod{v(d)}}} u_{des} \qquad \begin{array}{r} \forall d \in D,\ \forall e \in E, \\ \forall i \in \{1, \ldots, v(d)\}, \end{array} \tag{29}$$

*are valid for every optimal solution for the model DSL-BF.*

*Proof.* Suppose we have an optimal solution $u^*$ such that it violates at least one of the equalities (29). That means that there exists one arc $e \in E$, one demand $d \in D$ with $v(d) > 1$, and an index $i' = 2, \ldots, v(d)$, such that

$$\sum_{\substack{s \in S: \\ s \,\equiv\, i' \pmod{v(d)}}} u^*_{des} \neq \sum_{\substack{s \in S: \\ s \,\equiv\, i'-1 \pmod{v(d)}}} u^*_{des}. \tag{30}$$

If the arc $e$ is not used, then the equality is trivially satisfied, thus $d$ uses the arc $e$. Let $L = \{i', i' + v(d), \ldots, i' + k \cdot v(d)\}$ and $R = \{i' - 1, i' - 1 + v(d), \ldots, i' - 1 + h \cdot v(d)\}$ be the sets of the slots of variables of the left-hand side and the right-hand side of (30), respectively, with $k, h \in \mathcal{N}_0$. Since each pair of slots in L is at a distance of at least $v(d)$, then the demand $d$ can use at most one of them on $e$ so that $u^*$ is optimal. The same happens with the set R. Suppose $u^*_{des'} = 1$ with $s' \in L$. Since $d$ must use exactly $v(d)$ contiguous slots in $e$, then $d$ must also use slot $s' + 1$ or $s' - 1 \in R$. If $u^*_{des'} = u^*_{de,s'-1} = 1$, then the equality is satisfied, therefore assume $u^*_{des'} = u^*_{de,s'+1} = 1$. But, since $s' - 1 <$

$s' < s'+1 \leq s'+v(d)-1$, with the difference $(s'+v(d)-1)-(s'-1) = v(d)$, and both slots, i.e., $s'+v(d)-1$ and $s'-1$ belonging to $R$, then $d$ must use one of them, therefore the left-hand side and right-hand side of (30) equal 1. Due to symmetry we can follow the same reasoning when assuming $u^*_{des'} = 1$ for any $s' \in R$. Therefore, such an optimal solution $u^*$ does not exist and the equalities (29) are optimality equations. $\square$

Given the constraints of the model DSL-BF, a counterexample shows that the inequalities (27) and (28) are not implied by (29). Consider an instance of the RSA problem with $\bar{s} = 6$, and a graph with only one arc $e$ connecting the unique pair of nodes $i$ and $j$ which are the source and the destination of the unique demand $d$. If $v(d) = 3$, the vector $u'_{de*} = (\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2})$ satisfies the inequalities (29) besides the constraints of the model DSL-BF but not (27), since $u_{de3} < u_{de2}$, nor (28), since $u_{de4} < u_{de5}$, both for $i = 3$.

Using that $P(G,s,t)$ is composed by all the simple paths that connect $s$ with $t$, we can define a *minimal (s,t)-cut* to be a sub-set $C \subseteq E$ such that for every $P \in P(G,s,t)$ there exists $e \in C$ with $e \in P$, but for every $e' \in C$, $e' \neq e$, we have $e' \notin P$. An optimal solution $u^*$ must satisfy every demand using a path that cannot have cycles nor bifurcations, otherwise $u^*$ would not be optimal, and by definition each path between the source and the destination of $d$ has exactly one arc that belongs to $C$, in particular the paths used in the lightpath that satisfies $d$. This implies the following result.

**Proposition 4.1.** *For every demand $d \in D$ and every minimal $(s(d), t(d))$-cut $C$, the lightpath associated with $d$ in any optimal solution contains exactly one arc from $C$.*

**Theorem 4.3.** *Let $SC_d$ be the set of minimal $(s(d), t(d))$-cuts for a demand $d \in D$ in a digraph $G$, then the equalities*

$$\sum_{e \in C} \sum_{\substack{s \in S: \\ s \equiv i \pmod{v(d)}}} u_{des} = 1 \qquad \forall d \in D,\ \forall i \in \{1, \dots, v(d)\},\ \forall C \in SC_d, \tag{31}$$

*are valid for every optimal solution of $RSA(G, D, \bar{s})$.*

*Proof.* Consider an instance for RSA, a demand $d$ and a cut $C \in SC_d$. Given an optimal solution $u^*$, due to Proposition 4.1 there exists a unique arc $e' \in C$ such that $e'$ belongs to the path of the lightpath that satisfies the demand $d$ in $u^*$. Then $\sum_{s \in S} u^*_{de's} = v(d)$. Furthermore, $\sum_{s \in \hat{S}_i} u^*_{de's} = 1$ for every $i \in \{1, \dots, v(d)\}$, with $\hat{S}_i = \{s \in S : s \equiv i \pmod{v(d)}\}$. On the other hand, $\sum_{s \in S} u^*_{des} = 0$ for every $e \neq e'$ in $C$ by definition, so the equality (31) is satisfied by $u^*$. $\square$

**Corollary 4.1.** *The families (27), (29), and (31) are sufficient to ensure that in any solution that satisfies the model DSL-BF, each demand $d \in D$ uses either 0 or exactly $v(d)$ consecutive slots in every arc $e \in E$.*

**Corollary 4.2.** *The special cases of (31) given by $C = \delta^+(s(d))$ and $C = \delta^-(t(d))$ are called* contiguity IV *and* contiguity V *respectively.*

Note that when we use the cuts (29) we do not need to state the family of *contiguity IV* for every value of $i$ but only for one of them.

**Corollary 4.3.** *We also separate the sub-family obtained by taking only one sub-group of slots, for example by fixing $i = 1$. We call these families as* contiguity VI.

**Theorem 4.4.** *The equalities* contiguity-central VII*, restricted to every demand $d$ such that $2v(d) > \bar{s}$, namely,*

$$\sum_{e \in C} \sum_{s \in Y} u_{des} = |Y| \qquad Y = \{\bar{s} - v(d) + 1, \ldots, v(d)\},\ \forall C \in SC_d, \tag{32}$$

*and the special case when $C = \delta^+(s(d))$, called* contiguity-central VIII*, are optimality cuts for the model DSL-BF.*

*Proof.* Consider an instance of RSA and assume an optimal solution $u^*$ such that $u^*$ violates some equalities (32). There must exist a demand $d \in D$ such that $2v(d) > \bar{s}$, a non-empty set $C \in SC_d$, and $\sum_{e \in C} \sum_{s \in Y} u^*_{des} \neq |Y|$. Otherwise equalities (32) would be trivially satisfied. Since $u^*$ is optimal, due to Lemma 3.1, $d$ must be satisfied by a lightpath composed by a simple path $P$ and a channel with $v(d)$ consecutive slots. By definition of $C$, there must exist exactly one arc $e \in C$ such that $e \in P$, so $d$ must use $v(d)$ slots in that arc and no other slot in the rest of the arcs belonging to $C$, i.e., $\sum_{e' \in C \setminus \{e\}} u^*_{de's} = 0$. Due to integrality constraints and since $Y$ belongs to every posible interval of $v(d)$ consecutive slots in $[1, \ldots, \bar{s}]$, then $d$ must use every slot of $Y$ in the arc $e$, therefore $\sum_{s \in Y} u^*_{des} = |Y|$, a contradiction. ☐

The equalities (32) are not implied by the model given by (2)-(7) and the equalities (31).

**Corollary 4.4.** *We can disjoin the equalities (32) by slot, thus getting the set of equalities*

$$\sum_{e \in C} u_{des} = 1 \qquad Y = \{\bar{s} - v(d) + 1, \ldots, v(d)\},\ \forall s \in Y,\ \forall C \in SC_d, \tag{33}$$

*named* contiguity-central IX*, which are also valid for every demand $d$ such that $2v(d) > \bar{s}$. Analogously, we call as* contiguity-central X *the group of equalities resulting of taking $C = \delta^+(s(d))$.*

For the experiments, we implemented only this last group of equalities.

**Theorem 4.5.** *The inequalities* contiguity-position XI *and* contiguity-position XII*, i.e.,*

$$\sum_{s'=s+1}^{v(d)} u_{des'} \geq (v(d)-s)\ u_{des} \qquad \begin{array}{r}\forall e \in E,\, \forall d \in D,\\ \forall s \in \{1,\ldots,v(d)-1\}\end{array} \tag{34}$$

$$\sum_{s'=\hat{s}_d}^{s-1} u_{des'} \geq (s-\hat{s}_d)\ u_{des} \qquad \begin{array}{r}\forall d \in D,\, \forall s \in \{\bar{s}-v(d)+2,\ldots,\bar{s}\},\\ \forall e \in E,\ \hat{s}_d = \bar{s}-v(d)+1,\end{array} \tag{35}$$

*respectively, are implied by (27) and (28), respectively, therefore they are are optimality cuts for the model DSL-BF.*

*Proof.* Let $u'$ be a fractional solution for an instance of the model given by constraints (2)-(7) and the inequalities (28), let $d$ be a demand and $e$ an arc. Inequalities (28) force $u'_{des}$ for each $s$ higher than $\bar{s}-v(d)$ to be greater or equal than $u'_{de,s+1}$, and so $u'_{des}$ must to be greater than or equal to $u'_{des'}$ for every $s' > s$. Summing over every one of these new inequalities we have (35). The proof of the symmetric proposition is analogous. □

**Theorem 4.6.** *If we define $S' = \{1,\ldots,s-v(d)\} \cup \{s+v(d),\ldots,\bar{s}\}$, then, the inequalities* contiguity-distance XIII*, namely,*

$$\sum_{s'\in S'} u_{des'} \leq M(1-u_{des}) \qquad \forall e \in E,\, \forall d \in D,\, \forall s \in S, \tag{36}$$

*with $M = \min\{|S'|, v(d)\}$, are optimality cuts for the model DSL-BF.*

*Proof.* Let $u^*$ be an optimal solution for an instance of RSA such that there is a slot $s \in S$, an arc $e \in E$, and a demand $d \in D$ so that at least one inequality (36) is violated. Without loosing generality we can assume $v(d) \leq s \leq \bar{s}-v(d)+1$ and extend the following results to the border cases. If $u^*_{des} = 0$, as $M \geq |S'|$ we would have $\sum_{s\in S'} u^*_{des} > |S'|$, that cannot occur since $u^*_{des'} \in \{0,1\}$ for every $s' \in S$ due to the integrality constraints. Consequently, $u^*_{des}$ has to be equal to 1. But this, in addition to contiguity constraints, implies that another $v(d)-1$ slots must be used all around $s$, i.e., $\sum_{s'=s-v(d)+1}^{s+v(d)} u^*_{des'} = v(d)$, and since $S' = S - \{s-v(d)+1,\ldots,s+v(d)\}$, hence $\sum_{s'\in S'} u^*_{des'} = 0$ in order to be optimal. □

A counterexample shows that the optimality cuts (36) are not implied by the model constraints (2)-(7) together with (15), (29), and (31) with $C = \delta^+(s(d))$. Consider an instance of RSA with $\bar{s} = 12$ and only one demand $d$ with $v(d) = 2$. Then $M$ must be equal to $v(d)$. Let $u' \in \mathbb{R}^{|D||E|\bar{s}}$ be a vector such that $u'_{de*} = (\frac{1}{4},\frac{1}{4},\frac{1}{8},\frac{1}{8},0,\frac{1}{8},\frac{1}{2},\frac{1}{4},0,\frac{1}{8},\frac{1}{8},\frac{1}{8})$ for each $e$ all along a simple path between $s(d)$ and $t(d)$ and $u'_{des} = 0$ for the remaining slots and arcs. The vector $u'$ trivially satisfies the flow and the non-overlapping constraints and we can easily see that it does not violate the contiguity constraints. Likewise this fractional solution does not violate the inequalities (15), because the sum of every variable belonging to the arcs that outgo from $s(d)$ is exactly $v(d)$ and for the other vertices it is lower than or equal to $v(d)$. This solution also satisfies the equalities (29), because the sums of the two possible sets of slots are both equal to 1. However $u'$ violates the inequalities (36) on each arc of the path taking $s = 7$, i.e., the unique slot with value $\frac{1}{2}$.

**Theorem 4.7.** *The family of inequalities* contiguity-symmetrical XIV*, obtained by applying Theorem 2.1 to the model constraints (7), namely,*

$$\sum_{s'=1}^{v(d)} u_{des'} \geq v(d)\; u_{de1} \qquad \forall d \in D,\ \forall e \in E \tag{37}$$

$$\sum_{s'=s}^{f} u_{des'} \geq v(d)(u_{des} - u_{de,s-1}) \qquad \begin{array}{r}\forall d \in D,\ \forall e \in E,\ \forall s \in \{2, \ldots, \bar{s}\},\\ f = \min(\bar{s}, s + v(d) - 1),\end{array} \tag{38}$$

*are valid inequalities for the model DSL-BF.*

We can show that (37) and (38) are not implied by the model constraints, in particular by (7), by presenting a counterexample. Consider an instance of RSA with only one demand $d$ with $v(d) = 2$ and such that the arc $e = s(d), t(d)$ belongs to the graph. Assume $\bar{s} = 4$. Let $u'$ be a vector such that all of its elements are zero except for $u'_{de*} = (0, 1, \frac{1}{2}, \frac{1}{2})$. The vector $u'$ satisfies all constraints in particular (7) but not the inequalities (38) when taking $s = 2$, because $\frac{3}{2} < 2$. Generating another vector $u''$ by reversing $u'$, i.e., $u''_{de*} = (\frac{1}{2}, \frac{1}{2}, 1, 0)$, we can show the converse non-implication.

**Corollary 4.5.** *The variation to DSL-BF proposed in [44], i.e., DSL-ASCC, replaces constraints (6) and (7) with two families of inequalities. In particular, we can use*

$$u_{des_1} + u_{des_2} \leq u_{de(s_1+1)} + 1 \qquad \forall d \in D,\ \forall e \in E,\ \forall s_1, s_2 \in S,\ s_2 > s_1, \tag{39}$$

*as optimality cuts for DSL-BF. We call them as* contiguity-ASCC XV*.*

With a counterexample we can prove that (39) are not implied by the model constraints (2)-(7) and the inequalities *contiguity-symmetrical XIV*. Assume an instance of RSA with only one demand $d$ such that $v(d) = 3$, and only one arc $e$ connecting $s(d)$ with $t(d)$. Let $\bar{s} = 5$ and $u'$ be a vector such that $u'_{de*} = (\frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2})$. Vector $u'$ satisfies the non-overlapping and flow conservation constraints, and also satisfies both the contiguity constraints and the symmetrical (37)-(38) but $u'_{de2} - u'_{de3} + u'_{de4} > 1$. However, we believe that if we add flow inequalities (15), then (39) would be redundant.

## 5. Non-overlapping inequalities

In this section we present several valid inequalities, equations, and optimality cuts dealing mainly with the non-overlapping restriction, that is that no two demands can use the same slot within the same arc.

**Theorem 5.1.** *Given three slots $s_1, s_2, s_3 \in S$, inequalities*

$$u_{des_1} + \sum_{d' \in D\setminus\{d\}} u_{d'es_2} + u_{des_3} \leq 2 \qquad \forall d \in D,\ \forall e \in E,\ s_1 < s_2 < s_3, \tag{40}$$

*called* non-over I*, are optimality cuts for the model DSL-BF.*

*Proof.* Suppose there exists an optimal solution $u^*$ that violates any inequality (40), i.e., there exists $d \in D$, $e \in E$, and $1 \leq s_1 < s_2 < s_3 \leq \bar{s}$ such that $u^*_{des_1} + \sum_{d' \in D\setminus\{d\}} u^*_{d'es_2} + u^*_{des_3} > 2$. The solution $u^*$ must satisfy integrality constraints for each binary variable, and because of non-overlapping constraints $\sum_{d' \in D\setminus\{d\}} u^*_{d'es_2} \in \{0,1\}$, thus there must exist a particular demand $d' \neq d$ such that $u^*_{des_1} + u^*_{d'es_2} + u^*_{des_3} > 2$ and the three variables should be equal to 1. Since $u^*$ is optimal, each demand must use exactly $v(d)$ consecutive slots in $e$, so if $u^*_{des_3} = 1$ then there must exist a slot $s' \leq \bar{s} + 1$ such that (7) is satisfied with equality, i.e., $u^*_{des'} = 1$, $u^*_{de,s'+1} = 0$, and $\sum_{s \in A} u^*_{des} = v(d)$ with $A = \{s - v(d) + 1, s\}$, and since $|A| = v(d)$, therefore $u^*_{des} = 1$ for every $s \in A$. In addition, since $u^*$ is optimal, $d$ cannot use any slot not belonging to $A$ in $e$, hence, $s_1, s_3 \in A$ and also $s_2 \in A$ because this slot is between $s_1$ and $s_3$, thus $s_2$ must also be used by $d$ but, due to non-overlapping constraints (5), it cannot be because $s_2$ is used by $d'$. □

**Corollary 5.1.** *The inequalitites* non-over II*, namely,*

$$u_{d_2es_1} + u_{d_1es_2} + u_{d_2es_3} \leq 2 \qquad \begin{array}{l} \forall e \in E,\ \forall d_1, d_2 \in D,\ d_1 \neq d_2 \\ \forall s_1, s_2, s_3 \in S,\ s_1 < s_2 < s_3, \end{array} \tag{41}$$

*are satisfied by every point of the linear relaxation of the model DSL-BF that satisfies the inequalities (40).*

**Corollary 5.2.** *Inequalities (41) are optimality cuts for the model DSL-BF.*

**Corollary 5.3.** *As the number of inequalities in previous families may be too large to separate, in our experiments we also take $s_3 = s_2 + 1$, thus allowing $s_1$ and $s_2$ to be in the ranges $[1, \bar{s} - 2]$ and $[s_1 + 1, \bar{s} - 1]$, respectively. We call these sub-families as* non-over III *and* non-over IV *when reducing (41) and as* non-over V *and* non-over VI *when taking a sub-set of (40), respectively.*

**Definition 5.1.** *A set $D' \subseteq D$ is said to be a* minimal set of demands *if $\sum_{d \in D'} v(d) > \bar{s}$ and for every $d' \in D'$, $\sum_{d \in D'\setminus\{d'\}} v(d) \leq \bar{s}$,*

**Theorem 5.2.** *If $D' \subseteq D$ is a minimal set of demands then the inequalities* non-over-capacity VII*, namely,*

$$\sum_{s' \in S} u_{des'} \leq v(d) \sum_{d' \in D'\setminus\{d\}} \Big( v(d') - \sum_{s' \in S} u_{d'es'} \Big) \qquad \forall d \in D',\ \forall e \in E, \tag{42}$$

*are optimality cuts for the model DSL-BF.*

*Proof.* Let $u^*$ be an optimal solution for an RSA instance, and assume that $u^*$ violates at least one of the inequalities (42). Then there must exist a set $D' \subseteq D$, a demand $d \in D'$, and an arc $e \in E$ such that the inequality is not valid. Since $u^*$ is optimal, then each demand $d' \in D$ must use exactly 0 or $v(d')$ slots in $e$, and, given that $v(d') \in \mathbb{N}$, the sum of the right-hand side must be $k \cdot v(d)$ with $k \in \mathbb{N}_0$. But since the left-hand side must equal 0 or $v(d)$ due to the optimality of $u^*$, the only way to violate the inequality, is the right-hand side to be 0, i.e., each demand $d' \in D'$ included $d$ uses exactly $v(d')$ slots, which is not possible by definition of $D'$. $\square$

If we assume the constraints (2)-(7) and the family (23), we can prove that the inequalities (40) do not imply (42), nor vice versa.

**Theorem 5.3.** *If $D' \subseteq D$ is a minimal set of demands, the inequalities* non-over-capacity VIII*, namely,*

$$\sum_{s \in S} \sum_{d \in D'} u_{des} \leq \sum_{d \in D'} v(d) - \min_{d \in D'} \; v(d) \qquad \forall e \in E, \tag{43}$$

*are optimality cuts for the model DSL-BF.*

*Proof.* Assume there exists an optimal solution $u^*$ for an instance of the RSA such that for an $e \in E$ and a minimal set $D' \in D$ at least one inequality of the family (43) is violated. Since $u^*$ is optimal, each $d' \in D'$ must use 0 or $v(d')$ contiguous slots in $e$, but if any demand $d'$ does not use $e$, the sum of the slots used by the others is lower than or equal to the right side of the inequality, because $v(d') \geq \min_{d \in D'} v(d)$. Therefore all the demands in $D'$ use $e$, but due to non-overlapping, each slot in $e$ can be used by only one demand, and by definition of $D'$ we would need more than $\bar{s}$ slots. $\square$

If the constraints (2)-(7) hold, then the inequalities (42) do not imply (43).

**Corollary 5.4.** *In our computational experiments, we separate the case $|D'| = 2$ both in (42) and (43), which attains a better performance obtaining the families called* non-over-capacity IX *and* non-over-capacity X*, respectively.*

The following families of inequalities are motivated by the fact that the slot position used by one demand sometimes forces another demand not to be able to use a particular set of slots because its volume does not fit there without an overlap. This idea can be stated in many ways and generalized for a large structure giving us several families of inequalities.

**Definition 5.2.** *Let $e' \in E$ and $E' \subset E \setminus \{e'\}$. The set of arcs $PP = E' \cup \{e'\}$ is an* incoming private path *if and only if $e' \in P$ for every path $P = [e_1, e_2, ..., e_k] \subseteq E$ that contains two arcs $e_i$ and $e_j$, $1 \leq i < j \leq k$, with $e_i \notin PP$ and $e_j \in PP$.*

Analogously, we can define a similar structure when the path goes the other way, that is, from the inside of the private path to the outside.

**Definition 5.3.** *Let $e' \in E$ and $E' \subset E \backslash \{e'\}$. The set of arcs $PP = E' \cup \{e'\}$ is an* outgoing private path *if and only if $e' \in P$ for every path $P = [e_1, e_2, ..., e_k] \subseteq E$ that contains two arcs $e_i$ and $e_j$, $1 \leq i < j \leq k$, with $e_i \in PP$ and $e_j \notin PP$.*

In both cases of private paths we name the arc $e'$ as the *master arc*. When every path that connects an arc not in $PP$ with an arc in $PP$ must go through the master arc, we have an incoming private path, as shown in the example of Figure 1. Analogously, if every path that connects an arc in $PP$ with an arc not in $PP$ must use the master arc, then we have an outgoing private path.

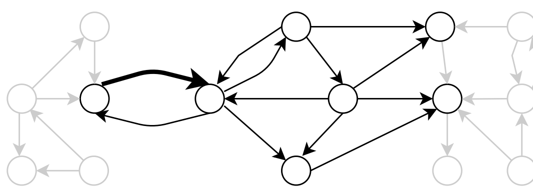

Figure 1: Example of private path. The black arcs form an incoming private path $PP$, because every path that connects an arc not in $PP$ with an arc in $PP$ must use the master arc $ab$.

**Definition 5.4.** *Given an incoming private path $PP$ with $ij$ as its master arc, we call as $D(PP)$ to the set composed by every demand $d \in D$ such that*

- *$s(d) \neq j$, and*
- *if $s(d) \in V(PP) \setminus \{i\}$ then $t(d) \in V(PP) \setminus \{i\}$ and does not exist any path with all its arcs in $PP \setminus \{ij\}$ connecting $s(d)$ with $t(d)$.*

*We define the analogous when $PP$ is an outgoing private path.*

That is, for every path $P$ that satisfies a demand $d \in D(PP)$, if $P$ uses some arc $e \in PP$, it must also use the arc $ij$.

**Lemma 5.1.** *Given a private path $PP$ with master arc $ij$, and a demand $d \in D(PP)$, if $d$ cannot use a slot $s$ in $ij$ then it cannot use $s$ in any arc $e \in PP$ without using a spurious cycle.*

*Proof.* Without loss of generality, assume $PP$ is an incoming private path. Suppose there exists such a demand $d \in D(PP)$. If $s(d) \notin V(PP)$, by definition of $PP$ there is no path $P \subseteq E \setminus \{e'\}$ that uses first an arc $e_1 \notin PP$ and then an arc $e_2 \in PP$, and if $s(d) \in PP$ by definition of $D(PP)$, then $t(d) \in V(PP)$ and there is no path $P \subseteq D(PP) \setminus \{e'\}$ connecting $s(d)$ with $t(d)$. Therefore, in both cases, if $d$ uses an arc belonging to $PP$ within a path $P$, $e'$ must also belong to $P$ and due to continuity if $d$ uses the slot $s$ in an arc belonging to $P$, $d$ must use this slot all along the path $P$, in particular in the master arc $e'$. $\square$

Since each arc in the instances that we use has its reverse, they have few private paths, so the following families were implemented by taking each arc of the graph as a private path (i.e., $PP = \{e\}$ for every $e \in E$, with $e$ as its master arc).

**Theorem 5.4.** *Let $PP$ be a private path with $e'$ its master arc. The inequalities* non-over-position XI*, namely,*

$$u_{d_1 e s_1} + \sum_{d' \neq d_1} u_{d' e' s} \leq 1 \qquad \begin{array}{l} \forall d_1 \in D(PP),\ \forall s \in \{2, \ldots, v(d_1)\}, \\ \forall s_1 \leq \max(s, \min_{d' \in D \setminus \{d_1\}}(v(d'))), \end{array} \tag{44}$$

*are valid for every $e \in PP$ for the model DSL-BF.*

*Proof.* Suppose $u'$ is a feasible solution for an instance of RSA that violates (44). That is, there exist a pair of slots $s \neq s_1 \in S$, a private path $PP$ with master arc $e'$, an arc $e \in PP$, and a demand $d_1 \in D(PP)$ such that the left-hand side of the inequality is greater than 1. Because of integrality and non-overlapping constraints, at most one demand may use the slot $s$ in the master arc $e'$. So both terms of the left-hand side must be equal to 1. Since $s \leq v(d_1)$, then $d_1$ cannot use the slots $1, \ldots, s$ in the master arc, due to the integrality, contiguity and non-overlapping constraints. If $s \geq \min_{d' \in D \setminus \{d_1\}} v(d')$ then $s_1 \leq s$, and in if $s < \min_{d' \in D \setminus \{d_1\}} v(d')$, then $d_1$ cannot use any slot in $s, \ldots, \min_{d' \in D \setminus \{d_1\}} v(d')$ in the master arc due to integrality, contiguity and non-overlapping constraints. Therefore, due to Lemma 5.1 $d_1$ cannot use $s_1$ in any arc belonging to $PP$. □

**Corollary 5.5.** *Given a private path $PP$ with master arc $e'$, the inequalities* non-over-position XII*, namely,*

$$u_{d_1 e s_1} + u_{d_2 e' s} \leq 1 \qquad \begin{array}{c} \forall e \in PP,\ \forall d_2 \in D,\ \forall d_1 \in D(PP),\ d_1 \neq d_2, \\ \forall s \in \{2, \ldots, v(d_1)\},\ \forall s_1 \leq \max(s, v(d_2)), \end{array} \tag{45}$$

*are satisfied by every point of the linear relaxation of the model DSL-BF that satisfies the inequalities (44).*

**Corollary 5.6.** *The inequalities (45) are valid for the model DSL-BF.*

**Theorem 5.5.** *The inequalities* non-over-position XIII*, namely,*

$$\sum_{s'=1}^{s_1} u_{d_1 e s'} \leq s_2 (1 - \sum_{d' \neq d_1} u_{d', e', s}) \qquad \begin{array}{c} \forall e \in PP,\ \forall d_1 \in D(PP), \\ \forall s \in \{1, \ldots, v(d_1)\}, \end{array} \tag{46}$$

*taking $s_1 = \max\{s, \min_{d' \in D \setminus \{d_1\}}(v(d'))\}$ and $s_2 = \min\{v(d_1), s_1\}$, and being $PP$ a private path with master arc $e'$, are optimality cuts for the model DSL-BF.*

*Proof.* Suppose $u^*$ is an optimal solution that violates at least one of the inequalities (46), i.e., there exist a private path $PP$, a demand $d_1 \in D(PP)$, an arc $e \in PP$ and a slot $s \leq v(d_1)$ such that the left-hand side of the inequality is strictly greater than the right-hand side. Due to the integrality and non-overlapping constraints the sum $\alpha = \sum_{d' \neq d_1} u^*_{d', e', s} = \{0, 1\}$ but as $s_2 \leq v(d_1)$ by definition and $\sum_{s' \in S} u^*_{d_1, e, s'} = \{0, v(d_1)\}$ by optimality of $u^*$, then $\alpha = 1$ in order to violate the inequality. That means, due to integrality and non-overlapping, that there exists a demand $d_2 \neq d_1$ that uses the slot $s$ on the master arc $e'$, and as $s \leq v(d_1)$ because of non-overlapping, integrality and contiguity, $d_1$ cannot use any slot lower than or equal to $s$ in $e'$, and due to the Lemma 5.1, in any arc $e \in PP$ neither. □

**Corollary 5.7.** *Since (46) implies* non-over-position XIV*, namely,*

$$\sum_{s'=1}^{s_1} u_{d_1 e s'} \leq s_2(1 - u_{d_2,e',s}) \qquad \begin{array}{r} s \in \{1, \ldots, v(d_1)\},\, \forall e \in PP, \\ \forall d_2 \in D,\, d_1 \in D(PP),\, d_1 \neq d_2, \end{array} \tag{47}$$

*where $PP$ is a private path and $e'$ its master arc, and $s_1 = \max\{s, v(d_2)\}$ and $s_2 = \min\{v(d_1), s_1\}$, then these inequalities are also optimality cuts for the model DSL-BF.*

**Definition 5.5.** *Let $PP$ be a private path and $s \in S$, then $D^{\geq}_s = \{d \in D(PP) : v(d) \geq s\}$ is defined to be the set of demands with volumes greater than or equal to s, and $D^{<} = D \backslash D^{\geq}$ is defined to be the remaining demands.*

**Theorem 5.6.** *Let $PP$ be a private path with $e'$ its master arc, then the variation of the inequalities (45), namely,*

$$\sum_{d' \in D^{\geq}_{s_2}} u_{d' e s_1} + \sum_{d' \in D^{<}_{s_2}} u_{d' e' s_2} \leq 1 \qquad \begin{array}{r} \forall e \in PP,\, s_1 \in \{1, \ldots, s_2 - 1\}, \\ s_2 \in \{2, \ldots, \max_{d \in D(PP)} v(d)\}, \end{array} \tag{48}$$

*called as* non-over-position XV *are valid inequalities for the model DSL-BF.*

*Proof.* Suppose that it is possible to find a feasible solution $u'$ violating at least one inequality (48) when taking a private path $PP$ with a master arc $e'$, a slot $s_2 \in \{2, \ldots, \max_{d \in D(PP)} v(d)\}$ and a slot $s_1 < s_2$. Because of the non-overlapping and integrality constraints each sum of the left-hand side of the inequality must equal 0 or 1, so to violate the equality both must equal 1, so there exists a demand $d_2 \in D^{<}_{s_2}$ that uses the slot $s_2$ in the master arc $e'$, and another demand $d_1 \in D^{\geq}_{s_2} \subseteq D(PP)$ which uses a slot $s_1 < s_2 \leq v(d_1)$ in any arc $e \in PP$. Since $d_1$ uses slot $s_1$ and $v(d_1) \geq s_2$, then $d_1$ must also use $s_2$: a contradiction since $u'$ is valid. $\square$

If we sum up inequalities (48) for every $s_1 \in \{1, \ldots, s_2\}$ we obtain the following family of optimality cuts.

**Corollary 5.8.** *The inequalities* non-over-position XVI*, namely*

$$\sum_{d' \in D^{\geq}_s} \sum_{s'=1}^{s} u_{d' e s'} \leq s\ (1 - \sum_{d' \in D^{<}_s} u_{d' e' s}) \qquad \forall s \leq \max_{d \in D(PP)} v(d), \tag{49}$$

*with $PP$ being a private path and $e'$ its master arc, are optimality cuts for every $e \in PP$ for the model DSL-BF.*

**Corollary 5.9.** *Due to Theorem (2.1) we can formulate the symmetrical for the inequalities (45)-(49), which are also valid for the model DSL-BF.*

**Theorem 5.7.** *Given a private path $PP$ with master arc $e'$, a demand $d_2 \in D(PP)$, and three slots $s_1, s_2, s_3 \in S$ such that $s_1 < s_2 < s_3$, and $s_3 - s_1 \leq v(d_2)$, then the family of inequalities*

$$\sum_{d' \neq d_2} u_{d',e',s_1} + u_{d_2 e s_2} + \sum_{d' \neq d_2} u_{d',e',s_3} \leq 2 \qquad \forall e \in PP, \tag{50}$$

*called as* non-over-position XVII*, are optimality cuts for the model DSL-BF.*

*Proof.* Given an instance of the RSA, assume it is possible to find an optimal solution $u^*$ that violates any of the inequalities (50). This implies that for some private path $PP$ with master arc $e'$, slots $s_1 < s_2 < s_3$, demand $d_2$ and arc $e$, the left-hand side of the inequality is greater than 2. Because of integrality and non-overlapping constraints each term of the left-hand side must equal 1 in order to violate the inequality, and $u^*_{d_2,e's_1} = u^*_{d_2,e's_3} = 0$. But, since the amount of slots between $s_1$ and $s_3$ is strictly less than the volume of $d_2$, then $e \neq e'$, and because of Lemma 5.1, $u^*_{d_2,es_2} = 0$ for every $e \in PP$. □

**Corollary 5.10.** *Let $PP$ be a private path and $e'$ its master arc. Let $d_1, d_3 \in D$, $d_2 \in D(PP)$, and $s_1, s_2, s_3 \in S$ be three different demands and slots such that $s_3 - s_1 \leq v(d_2)$ and $s_1 < s_2 < s_3$. Then, the inequalitites*

$$u_{d_1,e',s_1} + u_{d_2es_2} + u_{d_3,e',s_3} \leq 2 \qquad \forall e \in PP, \tag{51}$$

*are satisfied by every point of the linear relaxation of the model DSL-BF that satisfies the inequalities (50).*

**Corollary 5.11.** *The inequalities (51) are optimality cuts for the model DSL-BF.*

**Corollary 5.12.** *The previous family can be expressed by taking $M = s_3 - s_1 - 1$ and $2 \leq s_3 - s_1 \leq v(d_2)$ in the inequalities* non-over-position XVIII, *i.e.,*

$$\sum_{s'=s_1+1}^{s_3-1} u_{d_2es'} \leq M(2 - \sum_{d' \neq d_2} u_{d',e',s_1} - \sum_{d' \neq d_2} u_{d',e',s_3}) \qquad \forall e \in PP, \tag{52}$$

*that are also valid for the model DSL-BF.*

**Corollary 5.13.** *The inequalities generated this way are called* non-over-low XIX *and* non-over-low XX *when simplifying (45) and (44), respectively. The optimality cut (47) for this kind of private path is called* non-over-low XXI *when $s_1 = s2 = s$, and* non-over-low XXII *when $s_1 = \max\{s, v(d_2)\}$ and $s_2 = \min\{v(d_1), s_1\}$. Family (46) results in* non-over-low XXIII *while (48) results in the family* non-over-low XXIV. *Finally the inequalities (51)-(52) generate the families called* non-over-between XXV, non-over-between XXVI, *and* non-over-between XXVII, *respectively.*

Since the results of the experiments showed that no cut was added using *non-over-low XXIV*, we did not implement the reduced version of the inequalities (49).

**Corollary 5.14.** *The symmetrical families of* non-over-low XIX, *. . . ,* non-over-low XXIV *applied on the higher slots are called* non-over-high XXVIII, *. . . ,* non-over-high XXXIV.

**Theorem 5.8.** *The family* non-over-central-high XXXV, *given by the inequalities*

$$\sum_{s' \in Y} u_{des'} \geq |Y|\Big(\sum_{d' \in D \setminus \{d\}} u_{d'es_2} + u_{des_1} - 1\Big) \qquad \forall d \in D, \forall e \in E, \tag{53}$$

*where* $s_2 \in \{v(d)+1, \ldots, 2v(d)\}$*,* $s_1 < s_2$*, and* $Y = \{s_2 - v(d), \ldots, v(d)\}$*, are valid for the model DSL-BF.*

*Proof.* Suppose we have an instance of the RSA and assume there is a feasible solution $u'$ such that it does not satisfy at least one of the inequalities (53). Formally, there exist $d \in D, e \in E, s_1 < s_2 \in S$ with $s_2 \in \{v(d)+1, \ldots, 2v(d)\}$ such that the right-hand side of the inequality is strictly greater than the left-hand side. Due to non-overlapping and integrality constraints we have that the left-hand side is less than or equal to $|Y|$, and both $\sum_{d' \in D \setminus \{d\}} u_{d'es_2}$ and $u_{des_1}$ must equal 1, and thus $u_{des_2} = 0$. To get the left-hand side strictly less than $|Y|$, $d$ should not to use at least one slot $s \in Y$, because of integrality. Since $d$ uses slot $s_1$, $d$ must use $v(d)$ slots at the left or at the right of $s$ because of contiguity, but from the slot 1 to the highest slot of $Y$ and also from the first slot of $Y$ to $s_2 - 1 < 2v(d)$ there are exactly $v(d)$ slots, so there is no enough free slots on either side of $s$. $\square$

**Corollary 5.15.** *The family* non-over-central-low XXXVI *symmetrical to* non-over-central-high XXXV *is valid for the model DSL-BF.*

**Theorem 5.9.** *The inequalities called* non-over-central XXXVII*, given by*

$$\sum_{s' \in Y} u_{des'} \geq |Y| \Big( \sum_{d' \in D \setminus \{d\}} u_{d'es_1} + \sum_{d' \in D \setminus \{d\}} u_{d'es_3} + u_{des_2} - 2 \Big), \qquad (54)$$

*where* $s_1 \in \{1, \ldots, \bar{s} - 2v(d)\}$*,* $s_3 \in \{s_1 + v(d) + 1, \ldots, s_1 + 2v(d)\}$*,* $s_1 < s_2 < s_3$*, and* $Y = \{s_3 - v(d), \ldots, s_1 + v(d)\}$*, are valid for every* $d \in D$ *and* $e \in E$ *for the model DSL-BF.*

*Proof.* Analogous to the proof of Theorem 5.8. $\square$

## 6. Branch-and-cut algorithm

We have implemented a branch-and-cut procedure for RSA in order to evaluate the contribution of the families of valid inequalities, valid equations, and optimality cuts within a cutting plane environment. Each family is separated with an ad hoc separation procedure. The running time of these procedures is polynomial in almost all cases, being $\mathcal{O}((|D|^k |E| \bar{s}^r)$ with $k, r \leq 3$ the worst one. Procedures for the families *flow-dbrooms XVI*, *flow-dbrooms XVII*, *flow-cycles XVIII*$^*$, *flow-cycles XIX*$^*$, *flow-cycles XX*$^*$, *flow-cycles XXI*$^*$, *non-over-capacity VII*$^*$, *non-over-capacity VII*$^*$, *non-over-capacity IX*$^*$, *non-over-capacity X*$^*$, and *non-over-low XXIV* that involve complicated structures in the graph $G$, pre-compute the structures; and those families marked with an asterisk where the number of these structures is exponential, we pre-compute a large set of such structures and take a random sub-set each time.

Since almost every family of inequalities is polynomial, most of the implemented algorithms perform exhaustive searches. The e-companion to this manuscript (Online Resource 1) contains some details regarding the implementations of these algorithms with a few improvements and their complexities.

We define a parameter $\epsilon$ for each separation procedure in such a way that a violated inequality is added as a cut if the absolute value of the difference between the left-hand side and the right-hand side is at least $\epsilon$. The higher the value of this parameter, the fewer cuts added. This parameter is calibrated for each family and fixed at its best value for the final experiments. We evaluated different strategies for managing the separation procedures. In some of them, we employ an *effectiveness coefficient* $\varphi$, for each procedure, defined as the number of generated cuts divided by the number of calls to the procedure. When $\epsilon$ is calibrated, $\varphi$ gives us a way to compare the behavior of the families. The different strategies implemented are the following:

- *Brute Force* [BRF]: Execute all separation procedures.
- *Random* [RND]: Shuffle the list of procedures and iterate through it until at least one cut from $h$ different families is found, or until the end of the list is reached.
- *Most Effective* [EFF]: Sort the list of procedures by their effectiveness coefficient, and iterate through this list as in the *Random* strategy.
- *Most Effective With Random* [EFFR]: Iterate over the sorted list of procedures as in *Most Effective*, but randomly call one of the non-called procedures with a predefined probability.
- *Weighted Selection* [WTD]: Iterate over the sorted list of procedures, and execute each procedure with probability calculated as a function of its effectiveness coefficient.

The function used in strategy WTD is such that the resulting probabilities are guaranteed to be always greater than 0.05. The strategies EFF, and EFFR also contemplate the variation of pre-sorting the list of procedures according to the results of previous experiments, while the other strategies start with a random order. With the aim of forcing all the separation procedures to run at least once, the initial value for the coefficient $\varphi$ is infinite for every procedure in all the strategies that use it. Since this coefficient $\varphi$ –different for each family– depends on the cumulative number of generated cuts by procedure and the number of calls to each one of them, it is updated on every iteration resulting in higher value for those procedures that generate a greater amount of cuts per iteration. All the strategies, except BRF, have a parameter $h$ that indicates the amount of families to take cuts from. To keep the list of procedures updated, before returning from the callback, the algorithm iterates over those which were called and re-positions them. This process runs in time $\mathcal{O}(CT)$ being $C$ and $T$ the amount of called procedures and the total amount of them, respectively, which is negligible compared to the time required by the execution of the separation procedures.

## 7. Computational results

We now present our computational experiments. The implementation was performed within the Cplex 12.10 environment, and the experiments were carried out on a computer with an Intel(R) Xeon(TM) 2.80GHz CPU with 4 GB of RAM memory. For the codification Java SE 17.0 was utilized. Except for some experiments –that we explicitly mention– both for the branch-and-cut and for the branch-and-bound we turned off all Cplex primal heuristics, pre-solving and parallelization features.

The instance benchmark was generated using a script based on the literature, which is available at [47]. For the main set of 100 instances, nineteen real topologies were used, with $|V| \in \{6, \ldots, 43\}$ and $|E| \in \{9, \ldots, 176\}$. The number $\bar{s}$ of available slots depends on the instance, and it ranges from 5 to 200 in the smallest one and up to 150 in the others. The up to 236 demands are randomly generated with uniform distribution, with volumes ranging from 1 to 124 slots each. We separated a sub-set of 22 instances for the preliminary experiments, using 11 topologies, with up to 100 slots per arc, and 51 demands.

To compare the results we define a coefficient $\tau$ for each run as follows. Let $t$ be the running time in minutes, let $g \in [0, 1]$ be the optimality gap, and let $p = t/4$ be a penalty term. The coefficient $\tau$ is defined by

$$\tau = \begin{cases} t & \text{if the instance is solved within the time limit,} \\ t + p + g\ p & \text{if the instance is not solved within the time limit but} \\ & \text{a feasible solution is found,} \\ t + 2p & \text{if no feasible solution is found within the time limit.} \end{cases}$$

In this way, we penalize the lack of certainty when the time limit is reached, with a stronger penalty if no feasible solution was found. The lower the value of $\tau$, the better the result. For the following sections, each experiment is executed at least twice (three times for the parameter calibration) and the best result is selected. The total coefficient $\tau$ of multiple runs is calculated as the sum of the best coefficient for each particular run.

### 7.1. Selecting the most effective families

In order to calibrate the parameter $\epsilon$ for each procedure, we selected the sub-set of 22 instances. The execution time was limited to 4 minutes. For each family $f$ we experimented with $\epsilon \in \{0, 0.1, \ldots, M_f\}$, with $M_f$ a different value for each family $f$, so that almost no cuts are added (typically $M_f = 4$). For each such value of $\epsilon$ we calculated the coefficient $\tau$ for the best result of each instance.

| Family \ Coeff. $\epsilon$ | 0.0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|---|---|---|---|---|---|---|---|---|---|---|
| contiguity I | **6.99** | 9.38 | 13.14 | 8.98 | 15.27 | 18.27 | 23.59 | 36.45 | 30.75 | 31.36 |
| contiguity II | 9.58 | **8.25** | 15.62 | 18.39 | 15.97 | 17.01 | 15.67 | 20.54 | 22.37 | 22.32 |
| contiguity-symmetrical XIV | 20.11 | 21.20 | 18.00 | **17.80** | 18.07 | 24.63 | 23.47 | 20.51 | 20.90 | 22.87 |
| flow-branches XI | 27.90 | 28.04 | 24.21 | 26.57 | 31.08 | 25.60 | 24.44 | 31.02 | **19.59** | 29.49 |
| contiguity-distance XIII | 26.43 | 25.87 | 29.05 | **23.15** | 23.61 | 23.98 | 29.95 | 27.14 | 31.22 | 23.81 |
| contiguity V | 24.17 | 25.04 | 24.21 | 24.66 | 26.53 | 25.83 | **23.28** | 23.91 | 23.82 | 23.56 |
| contiguity IV | 25.34 | 25.00 | 24.77 | 23.78 | 25.77 | 24.24 | 25.06 | 27.05 | **23.56** | 25.09 |
| contiguity III | 25.57 | 34.98 | 26.61 | **23.86** | 31.36 | 30.97 | 25.62 | 37.81 | 39.13 | 23.90 |
| flow-volume VII | **26.00** | 27.65 | 49.47 | 73.94 | 53.16 | 26.74 | 45.48 | 27.05 | 28.45 | 31.34 |
| flow-branches X | 29.41 | 32.05 | 37.49 | **26.90** | 38.96 | 29.21 | 31.93 | 30.17 | 30.55 | 30.48 |
| contiguity VI | 38.43 | 32.15 | 27.62 | 36.98 | 33.86 | 35.53 | **27.35** | 29.66 | 35.32 | 46.84 |
| flow-volume VI | 36.56 | 36.45 | 36.23 | 37.37 | 37.33 | 37.57 | 43.42 | **33.28** | 39.17 | 36.99 |
| non-over-low XXII | 43.30 | 38.16 | 42.61 | 38.39 | 43.13 | 36.54 | 37.41 | 32.68 | **31.03** | 32.62 |
| contiguity-position XII | 34.33 | 35.94 | 34.51 | 36.85 | 35.69 | 35.51 | 36.70 | 36.56 | 33.00 | **29.31** |
| contiguity-ASCC XV | 38.47 | 33.86 | 37.00 | 34.21 | **29.45** | 38.74 | 38.14 | 39.20 | 41.94 | 43.25 |
| flow-dbrooms XVII | 43.56 | **32.18** | 35.29 | 35.41 | 35.78 | 35.86 | 36.23 | 36.58 | 38.71 | 40.89 |
| non-over-high XXVIII | 34.10 | 34.84 | 31.69 | 32.06 | **31.30** | 37.78 | 47.21 | 44.11 | 41.54 | 41.25 |
| flow-dbrooms XVI | 35.74 | 33.50 | 32.59 | 32.58 | 32.45 | 32.72 | 39.11 | 31.68 | **31.51** | 31.59 |
| contiguity-position XI | 41.77 | 37.62 | 43.94 | **35.05** | 46.03 | 36.71 | 45.11 | 44.24 | 43.91 | 37.38 |
| non-over-central XXXVII | 39.71 | 46.10 | 41.02 | 46.17 | 44.18 | 43.62 | 41.86 | 41.47 | **32.17** | 41.08 |
| flow-cycles XXIII | **33.60** | 33.92 | 35.10 | 35.85 | 36.06 | 39.23 | 34.32 | 34.95 | 35.18 | 37.75 |
| non-over-low XXI | 34.20 | 35.61 | **33.04** | 37.69 | 38.98 | 39.28 | 43.18 | 45.69 | 44.91 | 45.88 |
| flow-used-arcs XV | 34.34 | **33.17** | 36.91 | 36.22 | 40.56 | 38.07 | 39.84 | 37.68 | 33.61 | 37.13 |
| non-over-high XXVI | 38.54 | **33.70** | 43.32 | 36.77 | 46.67 | 37.49 | 44.39 | 40.47 | 48.80 | 47.36 |
| non-over-central-high XXXVI | 49.11 | 41.51 | 40.20 | 40.78 | 44.90 | 44.01 | 45.76 | 45.28 | **34.85** | 42.53 |
| flow-used-arcs XIV | 43.70 | 40.07 | 38.11 | 36.13 | 34.66 | 38.04 | 36.98 | 35.94 | 40.87 | **34.52** |
| non-over-between XXVII | 38.65 | **35.27** | 40.80 | 40.93 | 42.39 | 40.36 | 40.37 | 39.93 | 39.57 | 39.70 |
| flow-cycles XXIV | 36.78 | 36.39 | **35.28** | 40.71 | 39.05 | 43.94 | 44.35 | 41.04 | 40.79 | 43.99 |
| flow II | 46.50 | 36.54 | **36.15** | 41.15 | 42.48 | 43.31 | 42.86 | 41.68 | 42.61 | 40.94 |
| non-over-low XXIII | 42.83 | 41.69 | 37.30 | **35.51** | 43.22 | 41.71 | 39.43 | 36.74 | 36.45 | 36.35 |
| non-over-central-low XXXV | 40.90 | 46.92 | 41.65 | **36.17** | 44.88 | 41.58 | 46.13 | 42.97 | 38.97 | 43.54 |
| non-over VI | **36.06** | 39.18 | 43.62 | 41.07 | 42.33 | 42.10 | 44.89 | 39.17 | 40.37 | 45.06 |
| non-over-high XXIX | 36.36 | 41.09 | 39.05 | 38.59 | 38.61 | **36.10** | 40.82 | 44.37 | 43.44 | 43.34 |
| flow V | 42.52 | 42.62 | **36.78** | 41.47 | 40.32 | 41.74 | 42.01 | 41.79 | 44.09 | 43.27 |
| non-over-low XX | 37.88 | 40.69 | 44.85 | 40.43 | **37.00** | 37.13 | 41.78 | 37.10 | 43.18 | 42.98 |
| non-over-between XXVI | 42.17 | 41.03 | 46.44 | **37.56** | 38.96 | 46.87 | 48.27 | 43.46 | 42.09 | 42.83 |
| flow IV | 42.60 | 43.02 | **38.06** | 39.31 | 41.36 | 40.85 | 43.97 | 44.16 | 44.25 | 41.17 |
| non-over I | **38.23** | 47.89 | 39.29 | 44.88 | 41.48 | 39.88 | 40.99 | 39.08 | 40.12 | 38.48 |
| flow I | 42.85 | 43.71 | 43.68 | 43.69 | 38.47 | 38.25 | **38.23** | 38.57 | 40.94 | 40.97 |
| non-over-high XXVII | 39.90 | 41.07 | 41.55 | 41.15 | 38.32 | **38.27** | 40.54 | 45.65 | 44.70 | 44.99 |
| non-over V | 46.47 | 48.16 | 40.35 | 44.42 | 42.78 | **38.51** | 39.59 | 46.79 | 42.66 | 39.06 |
| flow III | 40.93 | 44.09 | 42.95 | 46.60 | 50.08 | 46.64 | 46.50 | **40.31** | 40.34 | 42.08 |
| flow-cycles XXII | 41.06 | 41.81 | 41.72 | 41.40 | 41.03 | 44.00 | 43.97 | 44.00 | 40.83 | **40.80** |
| non-over-high XXV | 51.09 | 48.11 | 49.03 | 48.34 | 58.36 | 50.15 | 46.46 | **42.18** | 51.03 | 62.02 |
| contiguity-central X | 47.50 | 47.50 | 48.58 | 48.54 | 48.55 | 48.85 | 45.67 | 48.82 | 42.51 | **42.21** |
| non-over-capacity IX | 44.28 | 43.61 | **42.60** | 42.61 | 42.65 | 42.67 | 42.62 | 42.63 | 42.63 | 42.67 |
| non-over-capacity X | 48.06 | 49.45 | 46.42 | 49.46 | 49.43 | 49.45 | **43.28** | 43.31 | 43.34 | 43.29 |
| flow-cycles XVIII | 51.67 | 51.56 | 48.05 | 50.28 | **44.30** | 47.28 | 52.45 | 46.15 | 47.20 | 49.25 |
| non-over IV | **43.73** | 52.38 | 58.02 | 57.26 | 47.49 | 50.19 | 50.21 | 47.92 | 52.61 | 52.60 |
| contiguity-central VIII | **44.38** | 47.49 | 48.54 | 48.50 | 48.49 | 48.51 | 48.51 | 48.51 | 48.53 | 47.93 |

Table 1: Performance coefficient $\tau$ for some of the best-performing separation procedures.

To measure the performance of each family according to each $\epsilon$ selected we report the sum of the coefficients $\tau$ over all the instances, resulting in one coefficient for each combination. Table 1 shows some of these results for some of the best-performing families. It is interesting to note that the best results were obtained with small values of $\epsilon$, suggesting that these cuts are indeed effective. We can observe a direct relation between the amount of cuts added by the algorithm and the effectiveness coefficient, and also the inverse relation between these and the coefficient $\tau$. These results support the choice of this coefficient of effectiveness as a parameter to measure the effectiveness of a family of cuts.

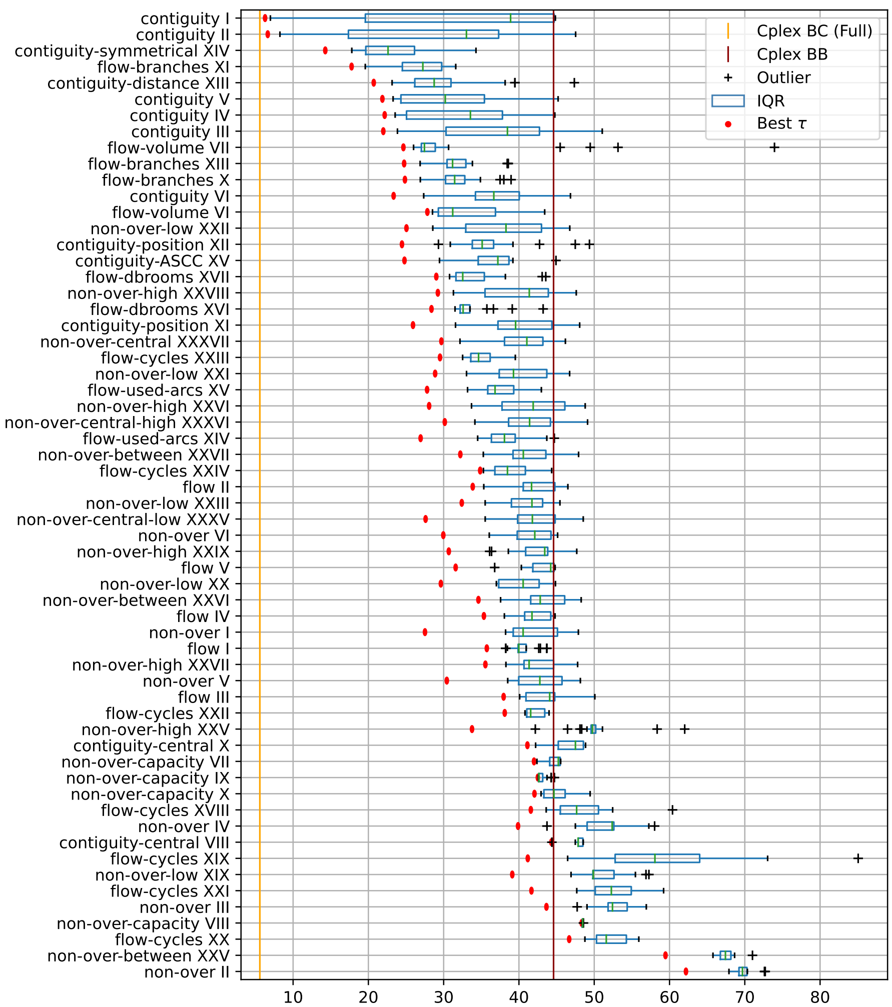


Figure 2: Comparison of the box-plots of the $\tau$ variation for each family of cuts according to the value of $\epsilon$. These values are also compared against the branch-and-bound and the full branch-and-cut implementations of Cplex. The figure also shows the best $\tau$ we could obtain for each family when selecting the best performing $\epsilon$ for each instance.

Table 2 shows additional results of some of the best-performing families, reporting for the best and the worst combination of $\epsilon$, the total coefficient $\tau$, the normalized number of generated cuts, and the normalized effectiveness coefficient $\varphi$, which, for each procedure $f$ and $\epsilon'$, is obtained as $100\ \frac{m_{\epsilon'}-l}{h-l}$, being $h$ and $l$ the highest and lowest values obtained in all the executions relative to $f$, and $m_{\epsilon'}$ the value obtained when $\epsilon = \epsilon'$. These normalized values are 34.06 and 33.67 for the amount of cuts and $\varphi$, respectively, in average for all the families when the best $\epsilon$ are selected, while they are 17.61 and 19.57, respectively, for

| Family name | Best combination $\epsilon$ | $\tau$ | % Cuts | % $\varphi$ | \|Cuts\| | Worst combination $\epsilon$ | $\tau$ | % Cuts | % $\varphi$ |
|---|---|---|---|---|---|---|---|---|---|
| contiguity I | 0.00 | 6.99 | 100.00 | 100.00 | 18253 | 4.00 | 44.84 | 0.00 | 0.00 |
| contiguity II | 0.10 | 8.25 | 70.21 | 71.16 | 14206 | 1.50 | 47.53 | 4.83 | 0.66 |
| contiguity-symmetrical XIV | 0.30 | 17.80 | 8.56 | 26.79 | 4427 | 1.90 | 34.29 | 3.27 | 5.18 |
| flow-branches XI | 0.80 | 19.59 | 0.96 | 0.82 | 4903 | 1.30 | 31.60 | 0.67 | 0.34 |
| contiguity-distance XIII | 0.30 | 23.15 | 36.31 | 55.53 | 52665 | 1.40 | 47.36 | 1.58 | 4.09 |
| contiguity V | 0.60 | 23.28 | 14.83 | 14.07 | 858 | 1.50 | 45.20 | 0.86 | 0.21 |
| contiguity VI | 0.80 | 23.56 | 7.78 | 7.34 | 608 | 4.00 | 44.77 | 0.00 | 0.00 |
| contiguity III | 0.30 | 23.86 | 15.96 | 17.82 | 3082 | 1.90 | 51.06 | 0.54 | 0.10 |
| flow-volume VII | 0.00 | 26.00 | 100.00 | 100.00 | 2193 | 0.30 | 73.94 | 0.28 | 1.86 |
| flow-branches XIII | 1.40 | 26.87 | 0.54 | 0.17 | 4328 | 4.00 | 38.59 | 0.00 | 0.00 |
| flow-branches X | 0.30 | 26.90 | 1.09 | 1.02 | 5820 | 0.40 | 38.96 | 0.91 | 1.27 |
| contiguity VI | 0.60 | 27.35 | 8.56 | 12.81 | 138 | 0.90 | 46.84 | 5.46 | 6.43 |
| flow-volume VI | 1.60 | 28.52 | 1.28 | 0.85 | 83 | 0.60 | 43.42 | 2.00 | 1.53 |
| non-over-low XXII | 1.10 | 28.56 | 2.41 | 14.42 | 256 | 4.00 | 46.75 | 0.00 | 0.00 |

Table 2: Total coefficient $\tau$, normalized number of generated cuts and normalized coefficient $\varphi$, for the best and worst performing $\epsilon$. The table also shows the total amount of cuts for the best performing $\epsilon$.

the worst $\epsilon$. The table also shows the total amount of cuts added for the best performing $\epsilon$.

We observed that the addition of almost every single separation procedure, with a proper parameter tuning, outperforms the simple branch-and-bound procedure, which attains a sum of the performance coefficient of 44.598. This also holds for many poorly-calibrated values of $\epsilon$. Figure 2 shows a comparison of the complete list of families against the generic branch-and-bound of Cplex and the *FullCplex*, i.e., the solver with generic cuts, the heuristics, pre-solve, re-start, and parallelization. Note that the coefficients obtained with our best separation procedures, although for $\epsilon$ very well calibrated, are comparable with the best configuration of Cplex, which attains a sum of the coefficients $\tau$ of 5.584.

### *7.2. Comparison of selection strategies*

We now report our experiments in order to evaluate the performance of the different selection strategies proposed in Section 6. In order to calibrate the parameter $h$, which limits the amount of different procedures to add cuts from, we used the same subgroup of 22 instances from the previous sub-section. We experimented with $h \in [5, 10, 15, 20, 25, 30]$. For each strategy and value of $h$, we considered the sum of the performance coefficient $\tau$ over all instances as a proxy for the overall performance. We also contemplate presorting or not the separation procedures according to previous experiments. The results are summarized in Table 3.

Once the best value of $h$ for each strategy was found, we experimented with the 100 instances with a time limit of 15 minutes. Our cuts were added to the generic branch-and-bound implemented by Cplex without pre-solve and primal heuristics. Therefore, the comparison is against this configuration, to show that our cuts are effective, and against the Cplex branch-and-cut to prove that they are better than the generic ones. Table 4 reports the total coefficient

| Strategy \ $h$ | 5 | 10 | 15 | 20 | 25 | 30 |
|---|---|---|---|---|---|---|
| MOST EFFECTIVE | 5.60 | 4.58 | 4.43 | 4.93 | 4.69 | 4.72 |
| MOST EFFECTIVE Pre-sorted | 3.78 | 4.46 | 4.99 | 5.10 | 4.59 | 5.51 |
| MOST EFFECTIVE RND | 5.36 | 4.32 | 4.59 | 5.23 | 4.79 | 6.97 |
| MOST EFFECTIVE RND Pre-sorted | 4.35 | 4.62 | 4.58 | 5.03 | 5.47 | 4.38 |
| RANDOM | 6.24 | 5.59 | 5.99 | 4.56 | 4.88 | 4.86 |
| WEIGHTED | 5.54 | 5.42 | 4.63 | 5.26 | 5.51 | 5.12 |
| WEIGHTED Pre-sorted | 5.66 | 6.06 | 5.49 | 5.21 | 5.19 | 5.23 |

Table 3: Performance of the selection strategies for different values for $h$ over the set of 22 instances.

| Strategy | $\tau$ | Optimal | Feasible | Unknown | Memory |
|---|---|---|---|---|---|
| Most Eff. with Rnd. 10 sorted | 1121.56 | 73 | 0 | 27 | 4 |
| Most Effective 5 sorted | 1237.84 | 68 | 1 | 31 | 4 |
| Most Eff. with Rnd. 5 sorted | 1273.96 | 68 | 0 | 32 | 4 |
| Most Effective 10 sorted | 1316.44 | 64 | 1 | 35 | 4 |
| Random 20 | 1350.32 | 64 | 0 | 36 | 3 |
| Weighted 15 | 1385.75 | 63 | 0 | 37 | 4 |
| Weighted 25 | 1393.53 | 63 | 0 | 37 | 4 |
| Brute Force | 1577.47 | 55 | 0 | 45 | 4 |
| CplexBC | 1858.67 | 40 | 10 | 50 | 4 |
| CplexBB | 2264.61 | 29 | 5 | 66 | 4 |

Table 4: Performance of some of the best selection strategies for the set of 100 instances with a time limit of 15 minutes.

$\tau$, the number of instances solved to optimality, the number of instances with feasible but not known as optimal solution, and the number of them where no feasible solution was found, discriminating those with memory problems. All these strategies outperform the two procedures implemented by Cplex. For the solved instances we improved all solution times with every strategy, whereas the best strategy was able to solve with optimality almost a third more instances than the branch-and-cut implementation of Cplex and almost twice as many as its branch-and-bound. For three of the four instances with memory problems we could not even build the model, therefore, it is necessary to use some pre-solver to handle them within a branch-and-bound or a branch-and-cut environment.

Except for one instance, the Cplex strategies were the only ones that returned non-optimal feasible solutions. Of those 14 instances, our worst strategy solved seven instances with optimality, while the best two solved 13 instances. There are only 27 instances that all the algorithms could solve with optimality within the time limit, and 39 solved by all algorithms except the branch-and-bound implementation of Cplex. For these instances, our best strategies proved optimality in half of the time needed by the branch-and-cut implementation of Cplex. Figure 3 summarizes these results.

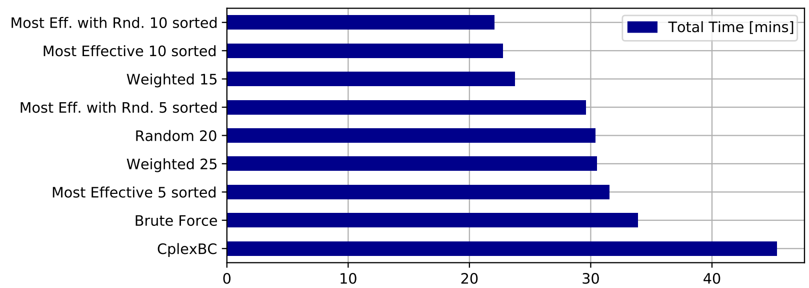


Figure 3: Comparison of the times needed to prove optimality for the 39 instances that all the strategies except the branch-and-bound could solve within a time limit of 15 minutes.

## 8. Conclusion

In the present work we explored several families of valid inequalities and equalities, and optimality cuts for the DSL-BF formulation. Since even characterizing the dimension of the polytope $RSA(G, D, \bar{s})$ is not straightforward, providing facetness results for the valid inequalities seems to be a challenging task. Instead, we showed that –together with the model constraints– the inequalities from some of these families do not imply each other. Meanwhile we moved forward with the experimentation in order to empirically determine their effectiveness. To that end, we implemented a branch-and-cut algorithm using these inequalities as cutting planes, but since there are more than 60 families, we proposed several filter and selection strategies in order to add only a few of them that improve the performance of the algorithm. Finally, we have presented the various experiments carried out in order to calibrate the different parameters of both the separation procedures and the selection strategies. These results suggest that the cuts are in fact effective, generally obtaining better results when the algorithm uses more of them. Likewise, the results seem to show that the vast majority of the families presented are, each one separately, sufficiently effective, not only to allow the branch-and-cut proposed to solve in less time and with better results the total of the instances tested compared to a generic branch-and-bound, but also to compete on these metrics against a branch-and-cut with generic cuts. The results of the comparison between the different strategies suggest that the proposed branch-and-cut has better performance than the generic branch-and-cut implemented by Cplex.

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