Facet-generating procedures for the maximum-impact coloring polytope

Mónica Braga[a], Javier Marenco[a,b]

[a] *Instituto de Ciencias, Universidad Nacional de General Sarmiento, Argentina*
[b] *Departamento de Computación, FCEyN, Universidad de Buenos Aires, Argentina*

Abstract

Given two graphs $G = (V, E_G)$ and $H = (V, E_H)$ over the same set of vertices and given a set of colors C, the *impact on* H of a coloring $c : V \to C$ of G, denoted $\mathcal{I}(c)$, is the number of edges $ij \in E_H$ such that $c(i) = c(j)$. In this setting, the *maximum-impact coloring* problem asks for a proper coloring c of G maximizing the impact $\mathcal{I}(c)$ on H. This problem naturally arises in the context of classroom allocation to courses, where it is desirable –but not mandatory– to assign lectures from the same course to the same classroom. In a previous work we identified several families of facet-inducing inequalities for a natural integer programming formulation of this problem. Most of these families were based on similar ideas, leading us to explore whether they can be expressed within a unified framework. In this work we tackle this issue, by presenting two procedures that construct valid inequalities from existing inequalities, based on extending individual colors to sets of colors and on extending edges of G to cliques in G, respectively. If the original inequality defines a facet and additional technical hypotheses are satisfied, then the obtained inequality also defines a facet. We show that these procedures can explain most of the inequalities presented in a previous work, we present a generic separation algorithm based on these procedures, and we report computational experiments showing that this approach is effective.

Keywords: coloring, integer programming, facet-generating procedures

1. Introduction

A recurring problem in course scheduling consists in determining which classrooms are to be assigned to each lecture of each course, in such a way that overlapping lectures receive different classrooms [5], where the starting and ending times of each lecture are given as part of the input. This situation is usually modeled by an undirected graph $G = (V, E_G)$, whose vertices represent the lectures and whose edges join pairs of lectures that cannot receive the same classroom since the corresponding time intervals have nonempty intersection, and by a set C of classrooms. The graph G is usually referred to as the *conflict graph* associated with the lectures. This problem corresponds to the classical vertex coloring problem, as any C-coloring c (i.e., an assignment $c : V \to C$ such that $c(i) \neq c(j)$ whenever $ij \in E_G$) corresponds to a feasible assignment of classrooms to lectures. This problem is feasible if and only if $|C| \geq \chi(G)$, where the *chromatic number* $\chi(G)$ represents the minimum number of colors in any feasible coloring of G.

A usual requirement in practical environments asks for all the lectures from the same course to be assigned to the same classroom. However, this requirement is not strict, and it can be violated if not enough classrooms are available. In order to take this requirement into account, the following combinatorial optimization problem was proposed in [1]. In addition to G, we have a second graph $H = (V, E_H)$ defined over the same set of vertices, in such a way that $ij \in E_H$ if and only if i and j are lectures from the same course. We assume $E_G \cap E_H = \emptyset$. If c is a coloring of G, we define the *impact of c on H* to be $\mathcal{I}(c) = |\{ij \in E_H : c(i) = c(j)\}|$, i.e., the number of edges from H whose endpoints receive the same color. Given two graphs $G = (V, E_G)$ and $H = (V, E_H)$ and a set C of colors, the *maximum-impact coloring* problem (MICP) consists in finding a C-coloring of G maximizing the impact on H. MICP is NP-hard [6], even when restricting G to be an interval graph and H to be the union of disjoint cliques [1], a situation usually arising in the context of classroom allocation.

Several formulations and techniques for the classical vertex coloring problem were applied in a wide range of applications (see, e.g., [3, 4, 7, 10]). Additionally, some of these applications have given place to scheduling software as in [8, 9].

Since integer programming techniques have been shown to be quite successful for the classical vertex coloring problem and for similar scheduling and timetabling problems, in [1] we proposed to tackle MICP with such techniques. We presented a natural integer programming formulation for MICP and identified several families of facet-inducing inequalities that turned out to be successful at enhancing the performance of a branch and cut procedure. Many of these families of valid inequalities are based on similar ideas and, consequently, the corresponding proofs of facetness contain repeated arguments. Moreover, similar ideas are present in the separation procedures associated with these families. These observations suggest the existence of general results explaining the facetness properties of the identified inequalities, and the design of a unified separation framework for them.

In this work we explore these issues, by presenting two validity- and facetness-preserving procedures that construct valid inequalities from existing inequalities, enlarging their supports (i.e., the set of indices with nonzero coefficientes in the inequality) in the process. We also introduce a generic separation algorithm based on these procedures, which starts from a set of inequalities with small supports and seeks to apply these procedures in order to obtain cuts with supports as large as possible. We present computational experiments suggesting that this approach may be competitive with respect to the application of individual separation algorithms for each family of valid inequalities.

This paper is organized as follows. Section 2 presents the integer programming formulation for MICP and states some known results on this formulation.

Section 3 presents the two procedures for constructing valid inequalities. Finally, Section 4 reports our computational experiments, and Section 5 includes concluding remarks and ideas for future work. The theoretical results contained in this work appeared without proofs in the conference paper [2].

2. Integer programming formulation

The following integer programming formulation for MICP, introduced in [1], is based on the *standard model* for vertex coloring. For $i \in V$ and $c \in C$, we define the binary *assignment variable* x_{ic} to be $x_{ic} = 1$ if the vertex i is assigned the color c, and $x_{ic} = 0$ otherwise. For every $ij \in E_H$ with $i < j$ we define the binary *impact variable* y_{ij} to be $y_{ij} = 0$ if the vertices i and j are assigned different colors. For $ij \in E_H$, $i < j$, we define $y_{ji} = y_{ij}$ as a notational convenience. In this setting, MICP can be formulated as follows.

$$
\begin{array}{llcll}
\max & \displaystyle\sum_{ij \in E_H, i<j} y_{ij} & & & \\
\text{s.t.} & \displaystyle\sum_{c \in C} x_{ic} & = & 1 & \forall i \in V \quad (1) \\
& x_{ic} + x_{jc} & \leq & 1 & \forall ij \in E_G,\ \forall c \in C \quad (2) \\
& y_{ij} & \leq & 1 + x_{ic} - x_{jc} & \forall ij \in E_H,\ i<j,\ \forall c \in C \quad (3) \\
& y_{ij} & \leq & 1 + x_{jc} - x_{ic} & \forall ij \in E_H,\ i<j,\ \forall c \in C \quad (4) \\
& x_{ic} & \in & \{0,1\} & \forall i \in V,\ c \in C \quad (5) \\
& y_{ij} & \in & \{0,1\} & \forall ij \in E_H,\ i<j \quad (6)
\end{array}
$$

The objective function asks for the total impact to be maximized. Constraints (1) and (2) ensure that the x-variables define a proper vertex coloring of G, whereas constraints (3) and (4) force $y_{ij} = 0$ if i and j receive different colors (if, e.g., $x_{jc} = 1$ and $x_{ic'} = 1$ for $c' \neq c$, then (3) implies $y_{ij} = 0$). We do not impose constraints forcing y_{ij} to take value 1 if i and j get the same color, since in any optimal solution this situation is guaranteed, and this property makes the resulting polytope much easier to study. Finally, constraints (5) and (6) ask the variables to be binary.

Definition 1 (maximum-impact coloring polytope). *Given two graphs $G = (V, E_G)$ and $H = (V, E_H)$ with $E_G \cap E_H = \emptyset$ and a finite set C, we define $\mathcal{P}_{MIC}(G, H, C) \subseteq \mathbb{R}^{|V||C|+|E_H|}$ to be the convex hull of the points $(x, y) \in \mathbb{R}^{|V||C|+|E_H|}$ satisfying constraints (1)-(6).*

The definition of $\mathcal{P}_{\text{MIC}}(G, H, C)$ implies Propositions 1 and 2, which will be used throughout this work. The converse implications do not hold in general, although they are true for most of the particular inequalities considered in this section. Similar assertions for facetness are also not true in general. For $S \subseteq V$, we define G_S to be the subgraph of G induced by S. If $\pi \in \mathbb{R}^{|V||C|}$ and $C' \subseteq C$, we define $\pi_{C'}$ to be the projection of π onto the entries associated with colors in C', i.e., $\pi_{C'} = (\pi_{ic})_{i\in V, c\in C'}$. For $S \subseteq V$, we define π_S to be the projection of π onto the entries associated with vertices in S, i.e., $\pi_S = (\pi_{ic})_{i\in S, c\in C}$. Similarly, if $\mu \in \mathbb{R}^{|E_H|}$ and $S \subseteq V$, we define $\mu_S = (\mu_{ij})_{ij\in E_H: i,j\in S}$.

Proposition 1. *If $\pi x + \mu y \leq \pi_0$ is valid for $\mathcal{P}_{MIC}(G, H, C\cup D)$ with $C\cap D = \emptyset$ and $\pi_{id} = 0$ for every $i \in V$ and $d \in D$, then $\pi_C x + \mu y \leq \pi_0$ is valid for $\mathcal{P}_{MIC}(G, H, C)$.*

Proposition 2. *If $\pi_S x + \mu_S y \leq \pi_0$ is valid for $\mathcal{P}_{MIC}(G_S, H_S, C)$ with $S \subseteq V$, $\pi_{ic} = 0$ for every $i \in V\backslash S$ and $c \in C$, and $\mu_{ij} = 0$ for every $ij \in E_H$ with $i \notin S$ or $j \notin S$, then $\pi x + \mu y \leq \pi_0$ is valid for $\mathcal{P}_{MIC}(G, H, C)$.*

Many families of facet-inducing inequalities for this polytope were presented in [1], some of which turned out to be quite effective within a cutting plane environment. We now summarize some of them, in order to point out the similarities that motivated the present work.

- Let $ij \in E_H$ and $D \subset C$ be a nonempty subset of colors. The following is the *partitioned inequality* associated with ij and D.

$$y_{ij} \;\leq\; 1 + \sum_{d\in D} x_{jd} - \sum_{d\in D} x_{id}. \tag{7}$$

- Let $K \subseteq V$ be a clique in G and let $i \in V\backslash K$ be a vertex such that $ik \in E_H$ for all $k \in K$. The following is the *vertex-clique inequality* associated with K and i.

$$\sum_{k\in K} y_{ik} \;\leq\; 1. \tag{8}$$

- Let $K \subseteq V$ be a clique in G and let $i \in V \setminus K$ such that $ik \in E_H$ for all $k \in K$. Let $D \subseteq C$ such that $|D| \leq |C| - |K|$. The following is the *clique-partitioned inequality* associated with the clique K, the vertex i, and the color set D.

$$\sum_{k\in K} y_{ik} \;\leq\; 1 + \sum_{d\in D}\sum_{k\in K} x_{kd} - \sum_{d\in D} x_{id}. \tag{9}$$

- Let $ij \in E_H$ and $k \in V$ such that $ik, jk \in E_G$, and let $c \in C$ and $D \subseteq C\backslash\{c\}$ with $D \neq \emptyset$. The following is the *semi-triangle inequality* associated with the vertex set $\{i, j, k\}$, the color c, and the color set D.

$$y_{ij} \;\leq\; 2 + (x_{jc} - x_{ic} - x_{kc}) + \sum_{d\in D}(x_{id} - x_{jd} + x_{kd}). \tag{10}$$

- Let $\{j, k, l\} \subseteq V$ be a triangle in G, and let $i \in V$ be a vertex such that $ij, ik \in E_H$ and $il \notin E_H$. Let $d_1, d_2 \in C$ with $d_1 \neq d_2$, and let $D \subseteq C\backslash\{d_1, d_2\}$ with $|D| \leq |C| - 3$. The following is the *semi-diamond* *inequality* associated with the triangle $\{j,k,l\}$, the vertex i, the colors d_1 and d_2, and the color subset D.

$$\begin{aligned} y_{ij} + 2y_{ik} &\leq 3 + \sum_{d \in D} (x_{kd} + x_{jd} - x_{id}) \\ &+ (x_{id_1} - x_{kd_1}) + (x_{kd_2} - x_{id_2}) - (x_{ld_1} + x_{ld_2}). \end{aligned} \quad (11)$$

- Let $K \subseteq V$ be a clique in G, and let $j \in K$ and $i \in V \setminus K$ be two vertices such that $ij \in E_H$. Let $D \subseteq C$ be a set of colors. The following is the *bounding inequality* associated with the clique K, the vertices i and j, and the color set D.

$$y_{ij} \leq \min\{|D|, |K|\} + 1 - \sum_{d \in D} \sum_{k \in K \setminus \{j\}} x_{kd} - \sum_{d \in D} x_{id}. \quad (12)$$

Within this list, the bounding inequalities are the only ones that do not define facets in general. However, they turned out to be effective for some instances in the computational experiments presented in [1].

Similar ideas appear throughout the inequalities in this list. Consider, e.g., the partitioned inequality (7), which asserts that y_{ij} must take value 0 if i is not assigned a color from D (hence the first summation in the RHS is null) and j is assigned a color from D (hence the second summation in the RHS takes value 1). The partitioned inequalities are facet-defining if $|C| \geq \chi(G)+1$, and provide a generalization of the model constraints $y_{ij} \leq 1 - x_{ic} + x_{jc}$, for $ij \in E_H$ and $c \in C$, by considering the set D of colors instead of a single color c (conversely, we can say that the model constraints (3) are a particular case of the partitioned inequalities, by taking $D = \{c\}$). This same construction appears in the inequalities (9)-(12). A similar situation holds for inequalities (8), (9), and (12), this time with the appearance of a clique within G in the inequalities, coupled with the idea that at most one vertex from the clique can receive a fixed color. Similar ideas appear in further families of valid inequalities presented in [1].

These repetitions cause the facetness proofs for these inequalities to be quite similar, and the separation procedures for these inequalities to contain replicated fragments of code. From a mathematical viewpoint, the appearance of the same idea more than once in several proofs suggests that there could be a unifying result explaining all these facets (or at least a common lemma that could be used in all these proofs), besides hindering the elegance of the proofs. From a computational viewpoint, the existence of similar code in different separation routines makes the code more difficult to maintain and more error-prone to create due to copying-and-pasting, besides the obvious elegance issues. These observations are the main motivations for the present work, namely we aim to unify these proofs within a single framework, which ideally could lead us to unified separation procedures. This can be achieved by so-called *facet-preserving procedures*, and it turns out that the two ideas mentioned in the previous paragraph can be formalized into such procedures. These ideas are explored in the next section.

3. Facet-preserving procedures

We introduce in this section the two validity- and facetness-preserving procedures for $\mathcal{P}_{\text{MIC}}(G, H, C)$. Both procedures take as input a valid inequality and produce a new inequality that is also valid and has a larger support than the original one. If, furthermore, the original inequality is facet-inducing, $|C| > \chi(G)$, and additional technical hypotheses are met, then this new inequality is also facet-inducing.

3.1. Procedure 1: Replacing a color by a set of colors

The first procedure takes as input a valid inequality for an instance (G, H, C) of the problem, such that the inequality involves at least one color $c \in C$ (i.e., there exists $i \in V$ such that the variable x_{ic} has nonzero coefficient in the inequality), and replaces all variables associated with this color by the variables associated with a set D of colors and the involved vertices. This new inequality is valid for the instance $(G, H, (C\backslash\{c\}) \cup D)$, namely the instance constructed by replacing the color c by the set D.

As defined previously for π, if $(x, y) \in \mathcal{P}_{\text{MIC}}(G, H, C \cup D)$ is a feasible solution, where C and D are two disjoint sets, we define x_C to be the projection of x onto the variables associated with colors in C, i.e., $x_C = (x_{ic})_{i\in V, c\in C}$. We say that the colors in C are *consecutive* if there is a linear ordering among them. This is satisfied if, e.g., $C = \{1, \ldots, |C|\}$. Finally, for $i \in V$, we define $N_G(i) = \{j \in V : ij \in E_G\}$ to be the set of neighbors of i in G. The set $N_H(i)$ is defined similarly for the graph H.

Procedure 1. *Let $\pi x + \mu y \leq \pi_0$ be a valid inequality for $\mathcal{P}_{MIC}(G, H, C)$. Fix $c \in C$, let D be a nonempty set of consecutive colors such that $C \cap D = \emptyset$, and define $C' = (C\backslash\{c\}) \cup D$. Define A and B to be the sets $A = \{i \in V : \pi_{ic} \neq 0\}$ and $B = \{i \in V : \mu_{ij} \neq 0$ for some $j \in N_H(i)\}$. Finally, for any feasible solution $(x, y) \in \mathcal{P}_{MIC}(G, H, C')$, define $I(x, y) = \{i \in B : y_{ij} = 1$ for some $j \in B \cap N_H(i)\}$. Assume that*

- *(i) for every $(x, y) \in \mathcal{P}_{MIC}(G, H, C') \cap \mathbb{Z}^{|V||C'|+|E_H|}$, $I(x, y)$ induces a stable set in G,*
- *(ii) for every $i \in A$, if there exist $(x, y) \in \mathcal{P}_{MIC}(G, H, C') \cap \mathbb{Z}^{|V||C'|+|E_H|}$ and a maximal stable set I' in G_B such that $I(x, y) \subseteq I'$ and $i \notin I'$, then $\pi_{ic} \leq 0$, and*
- *(iii) $\pi x_C + \mu y \leq \pi_0$ is valid for $\mathcal{P}_{MIC}(G, H, C \cup D)$.*

In this setting, the procedure generates the inequality

$$\sum_{i\in A} \sum_{d\in D} \pi_{ic} x_{id} + \sum_{i\in V} \sum_{d\in C\backslash\{c\}} \pi_{id} x_{id} + \sum_{ij\in E_H} \mu_{ij} y_{ij} \leq \pi_0, \tag{13}$$

for the instance (G, H, C').

The application of Procedure 1 to the model constraint $y_{ij} \leq 1 + x_{ic} - x_{jc}$, for $ij \in E_H$ and $c \in C$, provides the partitioned inequality (7), namely

$$y_{ij} \leq 1 + \sum_{d \in D} x_{id} - \sum_{d \in D} x_{jd},$$

when replacing c by the color set D. The variable x_{ic} is replaced by $\sum_{d \in D} x_{id}$, and the variable x_{jc} is replaced by $\sum_{d \in D} x_{jd}$. In this case we have $A = B = \{i, j\}$ with $ij \in E_H$ (so $ij \notin E_G$), hence the hypotheses (i) and (ii) of the procedure are trivially satisfied. The inequality (7) is valid for the extended instance with color set C', and the following result shows that this is the case in general when the hypotheses of Procedure 1 are satisfied.

Theorem 1. *If the hypotheses of Procedure 1 hold, then the inequality (13) is valid for* $\mathcal{P}_{MIC}(G, H, C')$.

Proof. Let $(x, y) \in \mathcal{P}_{\text{MIC}}(G, H, C') \cap \mathbb{Z}^{|V||C'|+|E_H|}$ be an arbitrary feasible solution of $\mathcal{P}_{\text{MIC}}(G, H, C')$. We shall show that (x, y) satisfies the inequality (13).

Let $I' \subseteq B$ be a maximal stable set in G_B such that $I(x, y) \subseteq I'$ (the hypothesis (i) ensures the existence of I'). Define also $M \subseteq I'$ to be the set of vertices in I' receiving a color from D in x, i.e., $M = \{i \in I' : \sum_{d \in D} x_{id} = 1\}$.

Construct a new solution $(x', y') \in \mathcal{P}_{\text{MIC}}(G, H, C \cup D)$ as follows. Set $x'_{ic} = 1$ for all $i \in M$, and keep in x' the color assigned in x to each vertex in $V \backslash M$, i.e.,

$$x'_{kt} = \begin{cases} 1 & \text{if } k \in M \text{ and } t = c, \\ 0 & \text{if } k \in M \text{ and } t \neq c, \\ x_{kt} & \text{if } k \notin M \text{ and } t \neq c, \\ 0 & \text{if } k \notin M \text{ and } t = c, \end{cases}$$

for $k \in V$ and $t \in C \cup D$. Also set $y'_{ij} = 0$ for $ij \in E_H$ with $i \in M$ and $j \notin M$ (or viceversa), and $y'_{ij} = y_{ij}$ otherwise.

We first show that this new solution is feasible. In order to verify that x' induces a proper coloring of $(G, C \cup D)$, we show that i and j are assigned different colors if $ij \in E_G$. To this end, we only need to consider the vertices in M (since the colors assigned to the remaining vertices are unchanged in the construction of x' from x). As $M \subseteq I'$ and I' is a stable set in G then there are no edges in G between pairs of vertices in M, hence no conflict is generated by assigning in x' the same color (namely, the color c) to a subset of vertices from I'.

It remains to verify that $y'_{ij} = 1$ implies that i and j are assigned the same color in x', for every $ij \in E_H$. Suppose, on the contrary, that there exists some $ij \in E_H$ such that $y'_{ij} = 1$ but i and j receive distinct colors in x'. By the construction of y', this implies that $y_{ij} = 1$, hence i and j are assigned the same color in x. Since i and j receive distinct colors in x' but the same color in x, we conclude that $i \in M$ and $j \notin M$ (or viceversa), a contradiction since in this case we set $y'_{ij} = 0$. We have, therefore, that $(x', y') \in \mathcal{P}_{\text{MIC}}(G, H, C \cup D)$, hence $\pi x'_C + \mu y' \leq \pi_0$ by the hypothesis (iii).

Call $LHS_1(x,y)$ to the LHS of (13) with the point (x,y). The following calculation shows that (x,y) satisfies (13).

$$
\begin{aligned}
LHS_1(x,y) &= \sum_{i\in V}\sum_{d\in C\setminus\{c\}} \pi_{id}x_{id} + \sum_{i\in A}\sum_{d\in D}\pi_{ic}x_{id} + \sum_{ij\in E_H}\mu_{ij}y_{ij} \\
&= \sum_{i\in V}\sum_{d\in C\setminus\{c\}} \pi_{id}x'_{id} + \sum_{i\in I'}\pi_{ic}x'_{ic} + \sum_{i\in A\setminus I'}\sum_{d\in D}\pi_{ic}x'_{id} + \sum_{ij\in E_H}\mu_{ij}y'_{ij} \\
&\leq \sum_{i\in V}\sum_{d\in C\setminus\{c\}} \pi_{id}x'_{id} + \sum_{i\in I'}\pi_{ic}x'_{ic} + \sum_{ij\in E_H}\mu_{ij}y'_{ij} \\
&= \pi_C x'_C + \mu y' \ \leq \ \pi_0.
\end{aligned}
$$

The second equality stems from the facts that (a) $x_{id} = x'_{id}$ for all $i \in V$ and $d \in C \setminus \{c\}$, (b) all the vertices in I' that receive a color in D in x, receive color c in x', (c) the x-variables corresponding to vertices in $A\setminus I'$ and colors in D remain unchanged in x', and (d) if $\mu_{ij} \neq 0$ and $y_{ij} = 1$, then i and j are both in $I(x,y)$ and share the same color in x, thus implying that $y'_{ij} = y_{ij}$. Each of these claims corresponds to each summation in the second expression. The first inequality stems from the fact that the hypothesis (ii) implies $\pi_{ic} \leq 0$ for every $i \in A \setminus I'$. The last inequality is implied by the hypothesis (iii), since $(x', y') \in \mathcal{P}_{\text{MIC}}(G, H, C \cup D)$. This shows that (x,y) satisfies (13) which is, therefore, valid for $\mathcal{P}_{\text{MIC}}(G, H, C')$. □

Although technical, the hypotheses (i)-(iii) are necessary for Theorem 1. Consider, e.g., the inequality

$$y_{ik} + y_{j\ell} \ \leq \ 3 - (x_{ic} + x_{jc}), \tag{14}$$

for $ik, j\ell \in E_H$ and $k\ell \in E_G$. This inequality asserts that if i and j are assigned the color c, then it cannot be the case that $y_{ik} = y_{j\ell} = 1$, since this would imply that k and ℓ receive color c, and this is not possible since $k\ell \in E_G$. Thus, this inequality is valid (although not facet-inducing in general). However, for the color c this inequality does not satisfy the hypothesis (i), as any solution (x,y) with $y_{ik} = y_{j\ell} = 1$ has $I(x,y) = \{i,j,k,\ell\}$, which is not a stable set in G. If we applied Procedure 1 to this inequality with color c, we would get

$$y_{ik} + y_{j\ell} \ \leq \ 3 - \sum_{d\in D}(x_{id} + x_{jd}),$$

which is not valid for $\mathcal{P}_{\text{MIC}}(G, H, C')$, as any solution with $x_{id} = x_{kd} = x_{jd'} = x_{\ell d'} = 1$ for $d, d' \in D$, $d \neq d'$, and $y_{ik} = y_{j\ell} = 1$ shows.

The hypothesis (ii) is also necessary for ensuring validity in Procedure 1. Consider, e.g., the simple semi-triangle inequality

$$y_{ij} \ \leq \ 2 + (x_{jc} - x_{ic} - x_{kc}) + (x_{ic'} - x_{jc'} + x_{kc'}) \tag{15}$$

for $ij \in E_H$ and $k \in V$ such that $ik, jk \in E_G$, and for $c, c' \in C$, $c \neq c'$. The inequality (10) corresponds to applying Procedure 1 for the color c', replacing

c' by the set $D \subseteq C$. However, we cannot apply Procedure 1 to (15) for the color c, since it does not satisfy the hypothesis (ii): since $I = B = \{i, j\}$ but $A = \{i, j, k\}$, then the hypothesis (ii) asks $\pi_{kc} \leq 0$, and this is not the case. Indeed, if we applied Procedure 1 to (15) for the color c, we would obtain the inequality

$$y_{ij} \;\leq\; 2 + \sum_{d \in D} (x_{jd} - x_{id} - x_{kd}) + (x_{ic'} - x_{jc'} + x_{kc'}), \qquad (16)$$

which is not valid for $\mathcal{P}_{\text{MIC}}(G, H, C')$, namely its RHS can take a negative value by assigning $x_{id} = x_{kd'} = 1$ for $d, d' \in D$, $d \neq d'$, and $x_{jc'} = 1$.

The hypothesis (iii) simply states that the original inequality $\pi x + \pi y \leq \pi_0$ remains valid when new colors are added to the instance (but the inequality remains unchanged). This is not always true, and may depend on the particular statement of the inequality. For example, let $ij \in E_G$ and consider the inequality $x_{ic} + x_{jc} \leq 1$. This inequality is valid for $\mathcal{P}_{\text{MIC}}(G, H, C)$ and is also valid for $\mathcal{P}_{\text{MIC}}(G, H, C \cup D)$. However, if we rewrite this inequality as $x_{ic} \leq \sum_{d \in C \setminus \{c\}} x_{jd}$ (by combining it with the model constraint (1)), then this new inequality is no longer valid for $\mathcal{P}_{\text{MIC}}(G, H, C \cup D)$. In this sense, the hypothesis (iii) is a way of ensuring that the initial inequality $\pi x + \pi y \leq \pi_0$ is expressed in such a way that adding colors to the instance does not affect its validity.

Hypotheses (i) and (ii) can be replaced by stronger statements that do not depend on checking conditions on feasible solutions. For example, if we ask $A \cup B$ to induce a stable set in G, then both hypotheses are satisfied. Alternatively, if (i') $G_{A \cup B}$ is composed by a clique K and an isolated vertex and (ii') $\pi_{ic} \leq 0$ for every $i \in K$, then also the hypotheses (i) and (ii) are satisfied. This observation gives rise to the following corollaries, which are used in Section 4 in order to identify valid inequalities that can be subjected to Procedure 1.

Corollary 1. *Assume the setting of Procedure 1. If (i') $A \cup B$ induces a stable set in G and (ii') $\pi x_C + \mu y \leq \pi_0$ is valid for $\mathcal{P}_{MIC}(G, H, C \cup D)$, then the inequality (13) is valid for $\mathcal{P}_{MIC}(G, H, C')$.*

Corollary 2. *Assume the setting of Procedure 1. If (i') $G_{A \cup B}$ is composed by a clique K and an isolated vertex, (ii') $\pi_{ic} \leq 0$ for every $i \in K$, and (iii) $\pi x_C + \mu y \leq \pi_0$ is valid for $\mathcal{P}_{MIC}(G, H, C \cup D)$, then the inequality (13) is valid for $\mathcal{P}_{MIC}(G, H, C')$.*

We now show that Procedure 1 also preserves facetness, namely if $\pi x + \pi y \leq \pi_0$ induces a facet of $\mathcal{P}_{\text{MIC}}(G, H, C)$ (and there are enough colors in order to characterize the dimension of this polytope) then (13) induces a facet of $\mathcal{P}_{\text{MIC}}(G, H, C')$. This result relies on the following fact.

Lemma 1 ([1]). *If $|C| > \chi(G)$, then $\mathcal{P}_{MIC}(G, H, C)$ has dimension $|V|(|C| - 1) + |E_H|$, and the model constraints (1) define a minimal equation system for this polytope.*

Define $\mathcal{A}(d) = \{x_{id}\}_{i\in V}$ to be the set of x-variables involving the color d, for any $d \in C$. Call $\dim(P)$ the dimension of a polytope P. Finally, let $d_0 = \min(D)$.

Theorem 2. *If the hypotheses of Procedure 1 hold and, furthermore,*

(a) $\mu \neq \mathbf{0}$,

(b) $\chi(G) < |C|$, *and*

(c) $\pi x + \mu y \leq \pi_0$ *induces a facet of* $\mathcal{P}_{MIC}(G, H, C)$,

then (13) induces a facet of $\mathcal{P}_{MIC}(G, H, C')$.

Proof. Let $F = \{(x, y) \in \mathcal{P}_{\text{MIC}}(G, H, C) : \pi x + \mu y = \pi_0\}$ be the facet of $\mathcal{P}_{\text{MIC}}(G, H, C)$ induced by $\pi x + \mu y \leq \pi_0$. Let $k = |V|(|C| - 1) + |E_H|$. Since $\chi(G) < |C|$, Lemma 1 implies $\dim(\mathcal{P}_{\text{MIC}}(G, H, C)) = k$, so $\dim(F) = k - 1$ and there exist k affinely independent points $(x^1, y^1), \dots, (x^k, y^k) \in F$. Construct k affinely independent points in the face of $\mathcal{P}_{\text{MIC}}(G, H, C')$ induced by (13) as follows. For $i = 1, \dots, k$, the solution $(\bar{x}^i, \bar{y}^i)$ is defined by $\bar{x}^i_{jd} = x^i_{jd}$ for $j \in V$ and $d \in C\backslash\{c\}$, $\bar{x}^i_{jd_0} = 1$ for every $j \in V$ such that $x^i_{jc} = 1$, and the other variables are set to 0 (i.e., vertices receiving color c in x^i are assigned color d_0 in $\bar{x}^i$, and the remaining vertices do not change). We also take $\bar{y}^i = y^i$. It is not difficult to verify that $(\bar{x}^i, \bar{y}^i)$ satisfies (13) with equality. Furthermore, these k constructed solutions are affinely independent, since $(x^1, y^1), \dots, (x^k, y^k)$ also are.

Consider now the projection of $(x^1, y^1), \dots, (x^k, y^k)$ onto the variables $\mathcal{A}(c) = \{x_{ic}\}_{i\in V}$. Since $\pi x + \mu y \leq \pi_0$ induces a facet of $\mathcal{P}_{\text{MIC}}(G, H, C)$, then the set of projected points must contain $|\mathcal{A}(c)| + 1$ affinely independent points (since otherwise there would exist an equation $\sum_{i\in V} \gamma_i x_{ic} = \gamma_0$ satisfied by all points in F, a contradiction since the model constraints (1) define a minimal equation system and $\mu \neq 0$), and assume w.l.o.g. that such projected points come from the solutions $(x^1, y^1), \dots, (x^t, y^t)$, with $t := |\mathcal{A}(c)| + 1$. For each color $d \in D\backslash\{d_0\}$, construct the set of solutions $(\bar{x}^{d1}, \bar{y}^{d1}), \dots, (\bar{x}^{dt}, \bar{y}^{dt})$ by setting $\bar{x}^{di}_{jc'} = x^i_{jc'}$ for $j \in V$ and $c' \in C\backslash\{c\}$, $\bar{x}^{di}_{jd} = 1$ for every $j \in V$ such that $x^i_{jc} = 1$, and the other variables are set to 0 (i.e., vertices receiving color c in x^i are assigned color d in $\bar{x}^{di}$, and the remaining vertices do not change). We also take $\bar{y}^{di} = y^i$. Since the projection of $(x^1, y^1), \dots, (x^t, y^t)$ onto the variables in $\mathcal{A}(c)$ is a set of affinely independent points, and the values of the variables in $\mathcal{A}(c)$ for these solutions coincide with the values of the variables in $\mathcal{A}(d)$ for the newly-constructed solutions, then $(\bar{x}^{d1}, \bar{y}^{d1}), \dots, (\bar{x}^{dt}, \bar{y}^{dt})$ also are affinely independent.

In order to complete the proof, we claim that the set $S := \{(\bar{x}^i, \bar{y}^i)\}_{i=1}^k \cup \{(\bar{x}^{dr}, \bar{y}^{dr})\}_{d\in D\backslash\{d_0\}, r\in\{1,\dots,t\}}$ has dimension $|V|(|C'|-1)+|E_H|-1$. To this end, take $(\lambda, \eta, \lambda_0)$ such that $\lambda\bar{x}^i + \eta\bar{y}^i = \lambda_0$ for $i = 1, \dots, k$ and $\lambda\bar{x}^{dr} + \eta\bar{y}^{dr} = \lambda_0$ for $d \in D\backslash\{d_0\}$ and $r = 1, \dots, t$. Define $\bar{C} := (C\backslash\{c\}) \cup \{d_0\}$. Since $\{(\bar{x}^i, \bar{y}^i)\}_{i=1}^k$ are affinely independent and have null values for the variables in $\mathcal{A}(d)$, for all

$d \in D\backslash\{d_0\}$, then $(\lambda_{\bar{C}}, \eta)$ is a multiple of (π, μ) and the coefficient vectors of the model constraints (1). For each $d \in D\backslash\{d_0\}$, the t equations

$$\begin{aligned} \lambda_{\{d\}} x^r_{\{c\}} \;=\; \lambda_{\{d\}} \bar{x}^{dr}_{\{d\}} \;&=\; \lambda_0 - \lambda_{\bar{C}\backslash\{d_0\}} \bar{x}^{dr}_{\bar{C}\backslash\{d_0\}} - \eta \bar{y}^{dr} \\ &=\; \lambda_0 - \lambda_{\bar{C}\backslash\{d_0\}} x^r_{C\backslash\{c\}} - \eta y^r \end{aligned}$$

for $r = 1, \ldots, t$ show that $\lambda_{\{d\}}$ is uniquely determined by $(\lambda_{\bar{C}\backslash\{d_0\}}, \eta, \lambda_0)$ and $\{(x^r, y^r)\}_{r=1}^t$, since $\{\bar{x}^{dr}_{\{d\}}\}_{r=1}^t = \{x^r_{\{c\}}\}_{r=1}^t$ are affinely independent and $\lambda_{\{d\}} \in \mathbb{R}^{t-1}$. Equations with the same coefficients hold for $\lambda_{\{d_0\}}$, namely

$$\begin{aligned} \lambda_{\{d_0\}} x^r_{\{c\}} \;=\; \lambda_{\{d_0\}} \bar{x}^{r}_{\{d_0\}} \;&=\; \lambda_0 - \lambda_{\bar{C}\backslash\{d_0\}} \bar{x}^{r}_{\bar{C}\backslash\{d_0\}} - \eta \bar{y}^{r} \\ &=\; \lambda_0 - \lambda_{\bar{C}\backslash\{d_0\}} x^r_{C\backslash\{c\}} - \eta y^r \end{aligned}$$

for $r = 1, \ldots, t$, showing that $\lambda_{\{d_0\}} = \lambda_{\{d\}}$ for every $d \in D\backslash\{d_0\}$. This implies that (λ, η) is a linear combination of the coefficient vectors of (13) and the model constraints (1), so the set $\{(\bar{x}^i, \bar{y}^i)\}_{i=1}^k \cup \{(\bar{x}^{dr}, \bar{y}^{dr})\}_{d \in D\backslash\{d_0\}, r \in \{1,\ldots,t\}}$ has dimension $|V|(|C'| - 1) + |E_H| - 1$.

This way, we construct a set of points in F with dimension $|V|(|C'| - 1) + |E_H| - 1$, thus showing that (13) induces a facet of $\mathcal{P}_{\mathrm{MIC}}(G, H, C')$. □

3.2. Procedure 2: Replacing a vertex by a clique

The second procedure generates valid inequalities from inequalities with smaller supports by replacing an edge of G by a clique in G. Specifically, if $ij \in E_G$ and $c \in C$, then we replace the variables x_{ic} and x_{jc} by the variables $\{x_{kc}\}_{k \in K}$, where K is a clique in G including i and j, and we perform a similar operation on the y-variables incident to i and j. This allows, e.g., to obtain the clique-partitioned inequalities (9) from the partitioned inequalities (7) when the vertices in K are twins.

We first provide some definitions. The vertices i and j are *true twins* in G if $ij \in E_G$ and $N_G(i) = N_G(j)$, and they are *false twins* if $ij \notin E_G$ and $N_G(i) = N_G(j)$. For $i \in V$ and $p \geq 1$, we define $G[i,p] = (V \cup \{i_1, \ldots, i_p\}, E'_G)$ to be the graph obtained from G by adding p new vertices (i.e., $i_1, \ldots, i_p \notin V$) in such a way that the vertices $i, i_1, \ldots i_p$ are true twins. In other words, E'_G is obtained by adding an edge between i_t and r, for every $r \in N_G(i) \cup \{i\}$ and $t = 1, \ldots, p$, and between the new vertices, i.e.,

$$\begin{aligned} E'_G \;=\; E_G \;&\cup\; \{i_t r : r \in N_G(i) \cup \{i\} \text{ and } t = 1, \ldots, p\} \\ &\cup\; \{i_t i_k : t, k \in \{1, \ldots, p\}, t \neq k\}. \end{aligned}$$

We also define $H(i,p) = (V \cup \{i_1, \ldots, i_p\}, E'_H)$ to be the graph obtained from H by adding p new vertices (i.e., $i_1, \ldots, i_p \notin V$) in such a way that the vertices $i, i_1, \ldots i_p$ are false twins. In other words, E'_H is obtained by adding an edge between i_t and r, for every $r \in N_H(i)$ and $t = 1, \ldots, p$, i.e.,

$$E'_H \;=\; E_H \;\cup\; \{i_t r : r \in N_H(i) \text{ and } t = 1, \ldots, p\}.$$

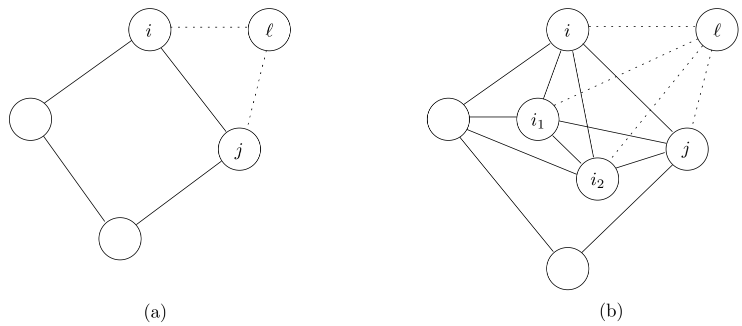

Figure 1: (a) Graphs G (solid edges) and H (dotted edges), and (b) graphs $G[i,p]$ (solid edges) and $H(i,p)$ (dotted edges), for $p = 2$.

Figure 1 shows an example of these constructions. Figure 1(a) shows the graphs G and H, by depicting the edges in E_G with solid lines and the edges in E_H with dotted lines (recall that $E_G \cap E_H = \emptyset$). Figure 1(b) shows the graphs $G[i,p]$ and $H(i,p)$ for $p = 2$ with the same solid/dotted notation for the edges. The clique $K = \{i, j, i_1, i_2\}$ is thus constructed in G, with the new vertices being true twins to i in G and false twins to i in H.

If $\pi x + \mu y \leq \pi_0$ is a valid inequality, define $V_{(\pi,\mu)}$ to be the set of vertices appearing in variables with nonzero coefficients in the inequality, i.e.,

$$\begin{aligned} V_{(\pi,\mu)} &:= \{i \in V : \pi_{ic} \neq 0 \text{ for some } c \in C\} \\ &\cup \{i \in V : \mu_{ij} \neq 0 \text{ for some } j \in N_H(i)\} \end{aligned}$$

We denote by $G_{(\pi,\mu)}$ the subgraph of G induced by $V_{(\pi,\mu)}$. Finally, we define $EH_{(\pi,\mu)} := \{ij \in E_H : \mu_{ij} \neq 0\}$.

Procedure 2. *Consider $\pi \in \mathbb{R}^{|V||C|}$ and $\mu \in \mathbb{R}^{|E_H|}$ such that the inequality*

$$\sum_{t \in V_{(\pi,\mu)}} \sum_{d \in C} \pi_{td} x_{td} + \sum_{tr \in EH_{(\pi,\mu)}} \mu_{tr} y_{tr} \leq \pi_0 \tag{17}$$

is valid for $\mathcal{P}_{MIC}(G_{(\pi,\mu)}, H_{(\pi,\mu)}, C)$. Fix $ij \in E_G$ and $c \in C$, and suppose there exists $\ell \in V$ such that $\ell \in N_H(i) \cap N_H(j)$. Assume that

(i) $\pi_{ic} = \pi_{jc}$ and $\pi_{id} = \pi_{jd} = 0$ for every $d \in C \backslash \{c\}$,

(ii) $\mu_{i\ell} = \mu_{j\ell}$, $\mu_{it} = 0$ for $t \in N_H(i) \backslash \{\ell\}$, and $\mu_{jt} = 0$ for $t \in N_H(j) \backslash \{\ell\}$, and

(iii) i and j are true twins in $G_{(\pi,\mu)}$ and false twins in $H_{(\pi,\mu)}$.

Define $K = \{i,j\} \cup \{i_1, \ldots, i_p\}$. In this setting, the procedure generates the inequality

$$\sum_{k \in K} \pi_{ic} x_{kc} + \sum_{t \in V \backslash \{i,j\}} \sum_{d \in C} \pi_{td} x_{td} + \sum_{k \in K} \mu_{i\ell} y_{k\ell} + \sum_{uv \in E_H \backslash \{i\ell, j\ell\}} \mu_{uv} y_{uv} \leq \pi_0, \tag{18}$$

for the instance $(G[i,p], H(i,p), C)$.

As an example, consider the inequality $y_{ij} + y_{ik} \leq 1$ for $i, j, k \in V$ such that $jk \in E_G$ and $ij, ik \in E_H$. This inequality asserts that it cannot be the case that $y_{ij} = y_{ik} = 1$, since this would imply that j and k are assigned the same color, a contradiction since $jk \in E_G$. The application of Procedure 2 to this inequality yields the vertex-clique inequality $\sum_{i \in K} y_{ik} \leq 1$, which is valid for the instance $(G[i,p], H(i,p), C)$, by taking $K = \{j,k\} \cup \{i_1, \ldots, i_p\}$.

Let $\bar{G} = (\bar{V}, \bar{E}_G)$ and $\bar{H} = (\bar{V}, \bar{E}_H)$ be any two graphs with the same vertex set, and $\bar{E}_G \cap \bar{E}_H = \emptyset$. Let $u, v \in \bar{V}$ be true twins in G and false twins in H. Given an integer solution $(x,y) \in \mathcal{P}_{\text{MIC}}(\bar{G}, \bar{H}, C)$, we define $\text{exc}_{uv}(x,y)$ to be the solution $(\bar{x}, \bar{y})$ for the instance $(\bar{G}, \bar{H}, C)$, obtained by exchanging the colors assigned to u and v and setting the y-variables accordingly, namely

$$\bar{x}_{wd} = \begin{cases} x_{ud} & \text{if } w = v, \\ x_{vd} & \text{if } w = u, \\ x_{wd} & \text{otherwise,} \end{cases}$$

for $w \in \bar{V}$ and $d \in C$. We also exchange the values of the variables corresponding to edges of $\bar{H}$ incident to u and v, i.e., $\bar{y}_{uw} = y_{vw}$ and $\bar{y}_{vw} = y_{uw}$ for every $w \in N_{\bar{H}}(u) = N_{\bar{H}}(v)$, and we keep the remaining y-variables unchanged. Since u and v are twins in both $\bar{G}$ and $\bar{H}$, then the constructed solution $(\bar{x}, \bar{y})$ is also feasible.

Theorem 3. *If the hypotheses of Procedure 2 hold, then the inequality (18) is valid for the polytope $\mathcal{P}_{MIC}(G[i,p], H(i,p), C)$.*

Proof. Let $(x,y) \in \mathcal{P}_{\text{MIC}}(G[i,p], H(i,p), C)$ be an arbitrary integer solution. We shall show that (x,y) satisfies (18). To this end, construct a feasible solution $(x', y') \in \mathcal{P}_{\text{MIC}}(G_{(\pi,\mu)}[i,p], H_{(\pi,\mu)}(i,p), C)$ by the following sequential procedure (i.e., we apply the following steps in sequence):

1. Let (x', y') be the restriction of (x,y) onto $G_{(\pi,\mu)}[i,p]$ and $H_{(\pi,\mu)}(i,p)$, namely $x'_{td} = x_{td}$ for $t \in V_{(\pi,\mu)} \cup \{i_1, \ldots, i_p\}$ and $d \in C$, and $y'_e = y_e$ for every edge e in $H_{(\pi,\mu)}(i,p)$.
2. If $x_{kc} = 1$ for some $k \in K \backslash \{i,j\}$ (at most one vertex from K can satisfy this property, since K is a clique in G), then define $(x', y') := \text{exc}_{ik}(x', y')$.
3. If $y'_{k\ell} = 1$ for some $k \in K \backslash \{i,j\}$ (again, at most one vertex from K can satisfy this property), then define $(x', y') := \text{exc}_{jk}(x', y')$.

Since i and j are twins in $G_{(\pi,\mu)}$ and $H_{(\pi,\mu)}$ by the hypothesis (iii), then the exchanges in Steps 2 and 3 do not affect the feasibility of the constructed solution. Since K is a clique in G, then at most one term in $\sum_{k \in K} \pi_{ic} x_{kc}$ is not null, and such a contribution is assigned to the vertex i in (x', y'). Similarly, at most one vertex from K can contribute to $\sum_{k \in K} \mu_{i\ell} y_{k\ell}$, and this contribution is assigned to either i or j in (x', y'). This implies that the LHS of (18) coincides for (x,y) and (x', y').

Define now $(x'', y'') \in \mathcal{P}_{\text{MIC}}(G_{(\pi,\mu)}, H_{(\pi,\mu)}, C)$ from (x', y') by projecting out the variables corresponding to the vertices $i_1, \ldots, i_p$. Then, $x''_{vd} = x'_{vd}$ for $v \in V_{(\pi,\mu)}$ and $d \in C$, and $y''_{uv} = y'_{uv}$ for every edge uv in $H_{(\pi,\mu)}$. This solution is feasible for the instance $(G_{(\pi,\mu)}, H_{(\pi,\mu)}, C)$ since it corresponds to removing the vertices $i_1, \dots, i_p$ in both graphs and keeping the color assignment in the remaining vertices. Furthermore, since $(x'', y'') \in \mathcal{P}_{\mathrm{MIC}}(G_{(\pi,\mu)}, H_{(\pi,\mu)}, C)$ then (x'', y'') satisfies (17).

Hypotheses (i) and (ii) imply that the LHS of (18) coincides for (x, y) and (x', y') and, furthermore, also coincides with $\pi x''_C + \mu y''$. This implies that the LHS of (18) for (x, y) is less than or equal to π_0, and (18) is satisfied. □

The hypothesis asking (17) to be valid for $\mathcal{P}_{\mathrm{MIC}}(G_{(\pi,\mu)}, H_{(\pi,\mu)}, C)$ is stronger than asking $\pi x+\mu y \le \pi_0$ to be valid for $\mathcal{P}_{\mathrm{MIC}}(G, H, C)$, which is a more natural hypothesis in this context. However, if we replace the former by the latter, then we must change hypothesis (iii) to ask i and j to be twins in G and H, a much stronger condition than being twins in $G_{(\pi,\mu)}$ and $H_{(\pi,\mu)}$, which would make it impossible to apply this procedure in most instances. It is frequent that inequalities involving an edge ij in G treat i and j symmetrically, and in this situation the hypothesis (iii) is usually satisfied.

A variant of Procedure 2 can be applied to inequalities coming from the classical vertex coloring polytope, i.e., inequalities $\pi x \le \pi_0$ not involving the y-variables, as follows.

Corollary 3. *Let $\pi x \le \pi_0$ be a valid inequality for $\mathcal{P}_{MIC}(G_{(\pi,\mathbf{0})}, H_{(\pi,\mathbf{0})}, C)$. Fix $ij \in E_G$ and $c \in C$ such that i and j are true twins in $G_{(\pi,\mathbf{0})}$, $\pi_{ic} = \pi_{jc}$, and $\pi_{id} = \pi_{jd} = 0$ for every $d \in C \backslash \{c\}$. Define $K = \{i, j\} \cup \{i_1, \dots, i_p\}$. Then, the inequality*

$$\sum_{k\in K} \pi_{ic} x_{kc} + \sum_{t\in V\backslash\{i,j\}} \sum_{d\in C} \pi_{td} x_{td} \ \le \pi_0 \tag{19}$$

is valid for the instance $(G[i, p], H(i, p), C)$.

Proof. Define $V' := V \cup \{\ell\}$, where $\ell \notin V$. Define $G' := (V', E_G)$ and $H' := (V', E'_H)$, where $E'_H := E_H \cup \{i\ell, j\ell\}$. The edge ij, the color c, the vertex ℓ, and the inequality $\pi x \le \pi_0$ satisfy the conditions of Theorem 3 for (G', H', C) (with $\mu = \mathbf{0}$), hence (18) is valid for the polytope $\mathcal{P}_{\mathrm{MIC}}(G[i, p], H(i, p), C)$. Since $\mu = \mathbf{0}$, the inequalities (18) and (19) coincide, and the result follows. □

Procedure 2 also preserves facetness if extra colors are available and additional hypotheses are satisfied. To this end, let $D := \{d_0, \dots, d_p\}$, where $D \cap C = \emptyset$, and assume that the inequality (18) is valid for $\mathcal{P}_{\mathrm{MIC}}(G[i, p], H(i, p), C \cup D)$ (i.e., it is expressed in such a way that it remains valid when new colors are added to the instance).

If $(x, y) \in \mathcal{P}_{\mathrm{MIC}}(G, H, C) \cap \mathbb{Z}^{|V||C|+|E_H|}$, we define the *extension* of (x, y) to the polytope $\mathcal{P}_{\mathrm{MIC}}(G[i, p], H(i, p), C \cup D)$ to be the solution $(\bar{x}, \bar{y})$ defined as

$$\bar{x}_{vr} \ = \left\{ \begin{array}{cl} x_{vr} & \text{if } v \in V \text{ and } r \in C, \\ 1 & \text{if there exists } t \in \{1, \dots, p\} \text{ s.t. } v = i_t \text{ and } r = d_t, \\ 0 & \text{otherwise,} \end{array} \right.$$

for $v \in V \cup \{i_1, \dots, i_p\}$ and $r \in C \cup D$, and by setting

$$\bar{y}_{uv} \ = \left\{ \begin{array}{cl} y_{uv} & \text{if } uv \in E_H, \\ 0 & \text{otherwise,} \end{array} \right.$$

for $uv \in E'_H$, i.e., the solution $(\bar{x}, \bar{y})$ corresponds to extending to $G[i,p]$ the coloring given by x, and by assigning the color d_t to the vertex i_t, for $t = 1, \dots, p$. We denote the extension of (x, y) by $\text{ext}(x, y)$. If (x, y) safisfies $\pi x + \mu y = \pi_0$, then $\text{ext}(x, y)$ satisfies (18) with equality too, since the vertices in $\{i_1, \dots, i_p\}$ are not assigned color c in $\text{ext}(x, y)$ and $y_{i_t \ell} = 0$ for $t = 1, \dots, p$, hence these vertices do not contribute to the LHS of (18).

Given an integer solution (x, y), a vertex $u \in V$, and a color $d \in C \cup D$, we define $\text{set}_{ud}(x, y)$ to be the vector $(\bar{x}, \bar{y})$ obtained by setting $\bar{x}_{ud} = 1$, $\bar{x}_{ud'} = 0$ for $d' \in (C \cup D) \setminus \{d\}$, and $y_{uv} = 0$ for $v \in N_H(u)$ with $x_{vd} = 0$, and leaving the remaining variables unchanged. This construction amounts to assigning color d to u, setting the y-variables associated with u accordingly. The vector $(\bar{x}, \bar{y})$ is not feasible if a neighbor of u in G is assigned color d in x.

Given an integer solution (x, y) and two twin vertices $u, v \in V$, recall that $\text{exc}_{uv}(x, y)$ is the solution $(\bar{x}, \bar{y})$ obtained by exchanging the colors assigned to u and v and setting the y-variables accordingly. Again, if (x, y) satisfies (18) with equality and $u, v \in K \setminus \{j\}$, then $\text{exc}_{uv}(x, y)$ also does.

We call an inequality to be a *trivial* inequality if only one variable has a nonzero coefficient in the inequality. Otherwise, we say that the inequality is *nontrivial.*

Theorem 4. *Let $D = \{d_0, \dots, d_p\}$, where $D \cap C = \emptyset$, and assume that the inequality (18) is valid for $\mathcal{P}_{MIC}(G[i,p], H(i,p), C \cup D)$. If the hypotheses of Procedure 2 hold and, furthermore,*

(a) $\chi(G) < |C|$,

(b) $N_H(i) = N_H(j) = \{\ell\}$,

(c) $\pi x + \mu y \leq \pi_0$ is nontrivial and induces a facet F of $\mathcal{P}_{MIC}(G, H, C)$,

(d) for every $t \in V \setminus \{i, j, \ell\}$ there exists a solution $(x, y) \in F$ with $x_{td} = 1$ for some $d \in C \setminus \{c\}$ with $\pi_{td} = 0$, and with $\mu_{ts} y_{ts} = 0$ for every $s \in N_H(t)$,

(e) there exists a solution $(x, y) \in F$ with $x_{id} = 1$ and $y_{i\ell} = 0$ (resp. $x_{jd} = 1$ and $y_{j\ell} = 0$), for some $d \in C \setminus \{c\}$, and

(f) there exists a solution $(x, y) \in F$ with $x_{id} = 1$ for some $d \in C \setminus \{c\}$ with $\pi_{\ell d} = 0$, $y_{i\ell} = 1$, and $\mu_{t\ell} y_{t\ell} = 0$ for every $t \in N_H(\ell) \setminus \{i\}$,

then the inequality (18) defines a facet of $\mathcal{P}_{MIC}(G[i,p], H(i,p), C \cup D)$.

Proof. To settle this result, we shall construct $k' := (|V| + p)(|C| + |D| - 1) + (|E_H| + p)$ affinely independent points satisfying (18) with equality. Since $|C| > \chi(G)$, then $|C \cup D| > \chi(G[i,p])$, hence $\mathcal{P}_{\text{MIC}}(G[i,p], H(i,p), C \cup D)$ has dimension k' and the existence of k' such affinely independent points will show that (18) induces a facet of this polytope.

Recall that F is the facet of the polytope $\mathcal{P}_{\text{MIC}}(G, H, C)$ induced by $\pi x + \mu y \leq \pi_0$, and let $(x^1, y^1), \dots, (x^k, y^k) \in F$ be $k := |V|(|C| - 1) + |E_H|$ affinely independent integer points in F. The points in $P^1 := \{\text{ext}(x^t, y^t)\}_{t=1}^k$ satisfy

(18) with equality and, furthermore, are affinely independent since their projections onto the variables in $\mathcal{P}_{\text{MIC}}(G, H, C)$ coincide with $(x^1, y^1), \ldots, (x^k, y^k)$. For $t = 1, \ldots, k$, call $(\bar{x}^t, \bar{y}^t) := \text{ext}(x^t, y^t)$.

For every $r \in \{1, \ldots, p\}$, construct the solution $\omega^r := \text{set}_{i_r d_0}(\bar{x}^1, \bar{y}^1)$, which assigns color d_0 to i_r and leaves the other vertices unchanged. Since no vertex is assigned color d_0 in $(\bar{x}^1, \bar{y}^1)$, then ω^r is feasible. Furthermore, since $\pi_{i_r d_r} = \pi_{i_r d_0} = 0$ and $\bar{y}^1_{i_r \ell} = 0$, then ω^r satisfies (18) with equality. Finally, ω^r is affinely independent w.r.t. the points in P^1, since this solution has $x_{i_r d_0} = 1$ but this variable is set to 0 in the solutions from P^1. This way, we construct a set $P^2 := \{\omega^r\}_{r=1}^p$ composed by p affinely independent solutions.

For every $r, s \in \{1, \ldots, p\}$, $r \neq s$, construct the solution $\omega^{rs} := \text{set}_{i_s d_r}(\omega^r)$, which assigns color d_0 to i_r, and color d_r to i_s. Again, since i_r and i_s are the only vertices in their color classes, then ω^{rs} is feasible, and $\pi_{i_r d_r} = \pi_{i_r d_0} = \pi_{i_s d_s} = \pi_{i_s d_r} = 0$ and $\bar{y}^1_{i_r \ell} = \bar{y}^1_{i_s \ell} = 0$ imply that ω^{rs} satisfies (18) with equality. Finally, ω^{rs} is affinely independent w.r.t. the points in $P^1 \cup P^2$, since this solution has $x_{i_s d_r} = 1$ but this variable is set to 0 in the previous solutions. This way, we construct a set $P^3 := \{\omega^{rs} : r, s = 1, \ldots, p, r \neq s\}$ composed by $p(p-1)$ affinely independent solutions.

Fix a vertex $t \in V \backslash \{i, j, \ell\}$ and a color index $r \in \{1, \ldots, p\}$. Let $(\hat{x}, \hat{y})$ be the solution specified by the hypothesis (d), i.e., a solution with $\hat{x}_{td} = 1$ for some $d \in C \backslash \{c\}$ with $\pi_{td} = 0$, and such that $\mu_{ts} \hat{y}_{ts} = 0$ for every $s \in N_H(t)$. Consider the solution $\omega^{tr} := \text{set}_{td_r}(\text{set}_{i_r d_0}(\text{ext}(\hat{x}, \hat{y})))$, i.e., a solution assigning color d_r to the vertex t, which in this solution is the only vertex with this color. Since $\pi_{td} = \pi_{td_r} = 0$ and $\mu_{ts} \hat{y}_{ts} = 0$ for every $s \in N_H(t)$, then t does not contribute to the LHS of (18), hence ω^{tr} satisfies (18) with equality. Finally, ω^{tr} is affinely independent with the previously-constructed solutions, since ω^{tr} has $x_{td_r} = 1$ but this variable takes null values in the previous solutions. With a similar construction we can get a solution with $x_{td_0} = 1$, thus getting a set P^4 with $(p+1)(|V| - 3)$ affinely independent points.

Let $(\hat{x}, \hat{y})$ be the solution specified by the hypothesis (e), i.e., a solution in F with $\hat{x}_{id} = 1$ for some $d \in C \backslash \{c\}$ and $\hat{y}_{i\ell} = 0$. For $r = 1, \ldots, p$, construct the solution $\omega^{ir} := \text{set}_{id_r}(\text{set}_{i_r d_0}(\text{ext}(\hat{x}, \hat{y})))$, which is feasible since no other vertex is assigned color i_r. Furthermore, since $\pi_{id} = 0$ by hypothesis (i) of Procedure 2, then this new solution also satisfies (18) with equality. Similarly to the previous constructions, ω^{ir} is affinely independent w.r.t. the points in $P^1 \cup \cdots \cup P^4$ since ω^{ir} has $x_{id_r} = 1$ but the previous constructions have $x_{id_r} = 0$. A similar construction allows us to construct a solution with $x_{id_0} = 1$. By repeating the argument with the vertex j, we construct a set P^5 with $2(p+1)$ affinely independent points.

Consider the solution $(\hat{x}, \hat{y})$ specified by the hypothesis (f), i.e., a solution in F with $\hat{x}_{id} = \hat{x}_{\ell d} = 1$ for some $d \in C \backslash \{c\}$ with $\pi_{\ell d} = 0$, and $\hat{y}_{i\ell} = 1$. Let $r \in \{1, \ldots, p\}$. Construct a feasible solution $\omega^{\ell r}$ from $(\hat{x}, \hat{y})$ by setting $x_{\ell d_r} = x_{id_r} = x_{i_r d_0} = 1$, keeping $y_{i\ell} = 1$, and leaving the remaining vertices and y-variables unchanged. We have $\pi_{\ell d} = 0$ and $\mu_{t\ell} y_{t\ell} = 0$ for every $t \in N_H(\ell) \backslash \{i\}$, hence $\omega^{\ell r}$ satisfies (18) with equality. Finally, $\omega^{\ell r}$ is affinely independent with the previously-constructed points, since $x_{\ell d_r} = 1$ in this solution but this variable takes value 0 in the previous constructions. A similar argument allows us to construct a point with $x_{\ell d_0} = 1$, so this construction provides a set P^6 with $p+1$ affinely independent points.

Consider again the solution $(\hat{x}, \hat{y})$ specified by the hypothesis (f). For $r = 1, \ldots, p$, construct the solution $\bar{\omega}^{ir}$ from $\text{ext}(\hat{x}, \hat{y})$ by setting $x_{\ell d_r} = y_{i_r \ell} = 1$, $y_{i\ell} = 0$, and leaving the remaining vertices and y-variables unchanged. This amounts to assigning color d_r to ℓ and setting the y-variables accordingly. Since $(\hat{x}, \hat{y}) \in F$, $\pi_{\ell d} = 0$, and $\mu_{t\ell} y_{t\ell} = 0$ for every $t \in N_H(\ell)\backslash\{i\}$, then the solution $\bar{\omega}^{ir}$ satisfies (18) with equality. Finally, $\bar{\omega}^{ir}$ is affinely independent w.r.t. the previously-constructed points, which have $y_{i_r \ell} = 0$. This way, we construct a set P^7 with p affinely independent points.

Let $r \in \{1, \ldots, p\}$ and $d \in C$, and consider a solution $(x, y) \in F$ with $x_{id} = 1$ (such a solution exists since $\pi x + \mu y \leq \pi_0$ defines a facet of $\mathcal{P}_{\text{MIC}}(G, H, C)$). Construct the solution $(\bar{x}, \bar{y}) := \text{exc}_{i,i_r}(\text{ext}(x, y)))$, namely a solution with $\bar{x}_{id_r} = \bar{x}_{i_r d} = 1$. Since the previously-constructed points have $x_{i_r d} = 0$ and this new solution has $x_{i_r d} = 1$, then this new solution is affinely independent w.r.t. the previous points. This way, we construct a set P^8 composed by $p|C|$ new affinely independent points.

The set $P^1 \cup \cdots \cup P^8$ thus contains $(|V| + p)(|C| + |D| - 1) + (|E_H| + p)$ affinely independent solutions satisfying (18) with equality, hence this inequality induces a facet of $\mathcal{P}_{\text{MIC}}(G[i, p], H(i, p), C \cup D)$. $\Box$

Some comments on the hypotheses in Theorem 4 are in order. The assumption asking (18) to be valid for $\mathcal{P}_{\text{MIC}}(G[i, p], H(i, p), C \cup D)$ plays a similar role to the hypothesis (iii) in Procedure 1, by supposing that the inequality remains valid when new colors are added. Since $\mathcal{P}_{\text{MIC}}(G, H, C)$ admits a nonempty equation system, then the inequalities can in principle be rewritten in such a way that the same expression is no longer a valid inequality if additional colors are added to the instance. This condition is satisfied for all the inequalities considered in this work.

The hypotheses (d) and (f) in Theorem 4 ask for the existence of particular solutions in F and may be difficult to check in general. However, they can be replaced by stronger conditions that may be easier to check in practice. If the hypothesis (c) holds, then the hypothesis (d) is trivially satisfied if $\pi_{td} = 0$ for every $d \in C\backslash\{c\}$ and $\mu_{ts} = 0$ for every $s \in N_H(t)$. Similarly (although more weakly), the hypothesis (f) is satisfied if $\pi_{\ell d} = 0$ for every $d \in C\backslash\{c\}$, $\mu_{t\ell} = 0$ for every $t \in N_H(\ell)\backslash\{i\}$, and there exists a solution $(x, y) \in F$ with $y_{il} = 1$ and $x_{ic} = 0$, namely a solution in which i and ℓ are assigned the same color and this color differs from c.

A slightly weaker statement than the hypothesis (e) is implied by the fact that $\pi x + \mu y \leq \pi_0$ induces a nontrivial facet F of $\mathcal{P}_{\text{MIC}}(G, H, C)$. Indeed, there must exist a solution $(x^1, y^1) \in F$ with $y^1_{i\ell} = 0$ (otherwise every point in F satisfies the equality $y_{i\ell} = 1$). Since $ij \in E_G$, such a solution has $x^1_{ic} + x^1_{jc} \leq 1$, and a similar argument shows that there exists a solution $(x^2, y^2) \in F$ with $y^2_{j\ell} = 0$ and $x^2_{ic} + x^2_{jc} \leq 1$. However, it may be the case that both $x^1_{jc} = x^2_{jc} = 1$, and this would hinder the construction of P^5 in the proof of Theorem 4. Due to this fact, we are forced to add the hypothesis (e) explicitly asking the existence of two such solutions with $x^1_{ic} = 0$ and $x^2_{jc} = 0$, respectively.

Although Procedure 2 asks $\pi x + \mu y \leq \pi_0$ to be valid for $\mathcal{P}_{\mathrm{MIC}}(G_{(\pi,\mu)}, H_{(\pi,\mu)}, C)$ (hence valid for $\mathcal{P}_{\mathrm{MIC}}(G, H, C)$), in Theorem 4 we need to ask this inequality to be facet-inducing for $\mathcal{P}_{\mathrm{MIC}}(G, H, C)$. Unfortunately, this asymmetrical situation seems to be unavoidable, as it is not clear how to extend the facetness property from $\mathcal{P}_{\mathrm{MIC}}(G_{(\pi,\mu)}, H_{(\pi,\mu)}, C)$ to $\mathcal{P}_{\mathrm{MIC}}(G, H, C)$ in general. Finally, note that Theorem 4 does not hold for trivial inequalities as, e.g., $x_{ic} \geq 0$ is facet-inducing for $\mathcal{P}_{\mathrm{MIC}}(G, H, C)$ but $\sum_{k \in K} x_{ic} \geq 0$ is not, for any clique $K \subseteq V$ in G with $|K| \geq 2$.

The two procedures can be applied iteratively. For example, the clique-partitioned inequality (9) is obtained by applying Procedure 1 and Procedure 2 to the inequality $x_{ic} + y_{ij} + y_{ik} \leq 1 + x_{jc} + x_{kc}$, for $ij, ik \in E_H$ and $jk \in E_G$, which is valid and facet-inducing if $|C| > \chi(G) + 1$. To this end, we first apply Procedure 2 in order to replace the edge jk by a clique K in G, thus obtaining

$$x_{ic} + \sum_{t \in K} y_{it} \;\leq\; 1 + \sum_{t \in K} x_{tc}. \tag{20}$$

We next apply Procedure 1 to (20) in order to replace the color c by the set D of colors. By combining the obtained inequality with the model constraint (1), we get the clique-partitioned inequality (9), which is thus valid and facet-inducing for $P(G[i,p], H(i,p), C \cup D)$ if $|C| + |D| > \chi(G) + p + 1$.

4. Computational experiments

Procedure 1 and Procedure 2 provide tools for constructing valid inequalities with potentially large supports starting from inequalities with small supports, and also preserve facetness when the right hypotheses are satisfied. This suggests a simple heuristic for trying to find violated valid inequalities within a cutting plane environment: start from a violated or "almost violated" small inequality, and then greedily try to enlarge the support of the inequality by using these procedures. If the resulting inequality is violated, then it can be added as a cut. In this section, we explore the design of a cut-generating computational procedure based on these ideas.

The separation procedure has a pool of generic valid inequalities for the polytope $\mathcal{P}_{\mathrm{MIC}}(G, H, C)$, which we propose to call *templates*. In this setting, templates are very simple inequalities with small supports, that are used as starting points of the search for cuts. In our implementation, we resort to the following pool of templates:

- T_1: the model constraint $y_{ij} \leq 1 + x_{ic} - x_{jc}$, for $ij \in E_H$ and $c \in C$,
- T_2: the *partitioned inequality* $y_{ij} \leq 1 + x_{ic} - x_{jc} + x_{id} - x_{jd}$, for $ij \in E_H$ and $c, d \in C$, $c \neq d$,
- T_3: the *edge inequality* $x_{ic} + x_{jc} \leq 1$, for $ij \in E_G$ and $c \in C$,
- T_4: the *vertex-clique inequality* $y_{ij} + y_{ik} \leq 1$ for $i, j, k \in V$ such that $jk \in E_G$ and $ij, ik \in E_H$,

- T_5: the *semi-triangle inequality* $y_{ij} \leq 2 + (x_{jc} - x_{ic}) + (x_{id} - x_{jd}) - (x_{kc} - x_{kd})$ for $i, j, k \in V$ such that $ik, jk \in E_G$ and $ij \in E_H$, and for $c, d \in C$, $c \neq d$,
- T_6: the *semi-diamond inequality* $y_{ij} + 2y_{ik} \leq 3 + (x_{kc} + x_{jc} - x_{ic}) + (x_{id} - x_{kd}) + (x_{ke} - x_{ie}) - (x_{\ell d} + x_{\ell e})$ for $i, j, k, \ell \in V$ such that $jk, j\ell, k\ell \in E_G$ and $ij, ik \in E_H$, and for $c, d, e \in D$, $c \neq d$, $c \neq e$, $d \neq e$,
- T_7: the *bounding inequality* $y_{ij} \leq 2 - (x_{ic} + x_{kc} + x_{\ell c})$ for $i, j, k, \ell \in V$ such that $jk, j\ell, k\ell \in E_G$ and $ij \in E_H$, and for $c \in C$.

Each template T specifies a set of vertices V^T and a set of colors C^T, as well as includes some constraints on the vertices given by pairs of vertices E_G^T that must be adjacent in G and pairs of vertices E_H^T that must be adjacent in H. For example, the template T_5 corresponding to the semi-triangle inequality has $V^{T_5} = \{i, j, k\}$ and $C^{T_5} = \{c, d\}$, with the additional constraints that $ik, jk \in E_G$ and $ij \in E_H$, i.e., $E_G^{T_5} = \{(i, k), (j, k)\}$ and $E_H^{T_5} = \{(i, j)\}$. In this context, i, j, and k are not pre-specified vertices of G, but are just symbolic identifiers instead, that will be associated with concrete vertices during the separation procedure.

For each template T, a subset $XC^T \subseteq C^T$ of the colors is defined to be the set of *expandable colors* (and these colors will be subjected to Procedure 1), and a subset $XE^T \subset E_G^T$ of the edges from G is defined to be the set of *expandable edges* (which will be subjected to Procedure 2). For each template T, we define the tuple

$$\mathbb{C}^T := < V^T, C^T, E_G^T, E_H^T, XC^T, XE^T >$$

to be the *configuration* associated with the template T. We say that a set of vertices $A \subseteq V$ and a set of colors $B \subseteq C$ is *compatible* with $\mathbb{C}^T$ if there exist bijections $v : V^T \to A$ and $w : C^T \to B$ such that $v(i)v(j) \in E_G$ for each $ij \in E_G^T$ and $v(i)v(j) \in E_H$ for each $ij \in E_H^T$.

In our implementation, we take $XE^T = E_G^T$ for $T \in \{T_3, T_4\}$, $XE^{T_7} = \{k\ell\}$, and $XE^T = \emptyset$ otherwise. We also take $XC^T = \{c\}$ for $T \in \{T_1, T_2\}$, $XC^{T_5} = \{d\}$, $XC^{T_6} = \{e\}$, and $XC^T = \emptyset$ otherwise. These definitions ensure that the procedures can be properly applied. Indeed, expandable colors and edges for each template are chosen in such a way that the hypotheses of each procedure are satisfied, and this is simple to check for each of them. In particular, these inequalities are not only valid for $\mathcal{P}_{\text{MIC}}(G, H, C)$, but they are also valid for $\mathcal{P}_{\text{MIC}}(G_{(\pi,\mu)}, H_{(\pi,\mu)}, C)$ (as required by Procedure 2), since the validity theorems for them do not ask for any conditions for the vertices with null coefficients.

Given a fractional solution $(x^*, y^*) \in \mathcal{P}_{\text{MIC}}(G, H, C)$, the separation procedure first detects all violated and "almost violated" instances of these templates. We take into account the inequality $\pi x + \mu y \leq \pi_0$ if $\pi x^* + \mu y^* \geq \pi_0 + \varepsilon$, for some small (usually negative) ε, and we have used $\varepsilon = -0.25$ in our experiments. To this end, for each template T, all sets $A \subseteq V$ and $B \subseteq C$ that are compatible with $\mathbb{C}^T$ are identified, and all detected inequalities thus generated are stored. The search for all subsets of vertices and colors compatible with each template configuration is performed by a backtracking procedure, in order not to continue the search when the current assignment cannot be extended to a compatible assignment. Since the templates involve small supports, such a bracktracking procedure is not computationally expensive.

Each valid inequality thus found is then subjected to Procedures 1 and 2. We greedily apply these procedures in order to enlarge the support of the obtained inequalities. We first try to apply Procedure 1 for each expandable color in XC^T, by defining D to be the largest set of colors increasing the LHS of the inequality. The contribution of each color to the LHS of the constructed inequality is independent of the other colors in D, so a set D with maximum cardinality can be obtained by including in D the colors with a positive contribution to the LHS. We then apply Procedure 2 for each expandable edge $e \in XE^T$, by greedily identifying a clique in G including e that enlarges the LHS of the inequality.

The theoretical formulation of both procedures generates a valid inequality for a modified instance of the problem, namely the instance with color set $(C\backslash\{c\}) \cup D$ in Procedure 1 and the instance $(G[i,p], H(i,p), C \cup D)$ in Procedure 2. However, in our implementation we keep the instance fixed and execute the procedures with properly-constructed sub-instances of the original instance, as follows. Recall that $V_{(\pi,\mu)}$ is the set of vertices appearing as indices of the variables with nonzero coefficients in $\pi x + \mu y \leq \pi_0$, and also define $C_\pi \subseteq C$ to be the set of colors appearing in x-variables with nonzero coefficients in π. We apply Procedure 1 to the inequality $\pi x + \mu y \leq \pi_0$ and a color $c \in C_\pi$, which is possible since validity for $\mathcal{P}_{\mathrm{MIC}}(G,H,C)$ implies validity for $\mathcal{P}_{\mathrm{MIC}}(G,C,C_\pi)$. In this setting, we select some color $c' \notin C_\pi$ and replicate the x-variables involving c, namely we replace the original inequality

$$\sum_{d \in C_\pi} \sum_{i \in V} \pi_{id} x_{id} + \sum_{e \in E_H} \mu_e y_e \ \leq \ \pi_0 \tag{21}$$

by the extended inequality

$$\sum_{d \in C_\pi} \sum_{i \in V} \pi_{id} x_{id} + \sum_{i \in V} \pi_{ic} x_{ic'} + \sum_{e \in E_H} \mu_e y_e \ \leq \ \pi_0. \tag{22}$$

In the theoretical formulation of Procedure 1, this corresponds to taking $C = C_\pi$, i.e., the set of colors present in the support of the inequality, and taking $D = \{c', c''\}$, where $c', c'' \notin C_\pi$ and replacing color c'' by c in the resulting inequality. Since the colors are indistinguishable, such a replacement does not impact the validity of the inequality. We iteratively perform this step for every color $c' \notin C_\pi$ such that the LHS of (22) evaluated at (x^*, y^*) is larger than the LHS of (21) evaluated at (x^*, y^*). Note that the iterative application of Procedure 1 with colors $d_1, \ldots, d_t$ amounts to applying Procedure 1 once for the set $D = \{d_1, \ldots, d_t\}$, so the final result amounts to applying this procedure with a potentially large set of colors.

Similarly, when applying Procedure 2, we take an edge $ij \in E_G$ with $i, j \in V_{(\pi,\mu)}$ and select a vertex $k \notin V_{(\pi,\mu)}$ such that $ik, jk \in E_G$, and replicate the variables involving i or j by k, namely by replacing the original inequality

$\pi x + \mu y \leq \pi_0$ by

$$\pi x + \mu y + \sum_{c \in C} \pi_{ic} x_{kc} + \sum_{t \in N_H(i)} \mu_{it} y_{kt} \ \leq \pi_0.$$

In this inequality, the variables associated with the vertex k take the same coefficient as the variables associated with the vertex i, namely x_{kc} is multiplied by π_{ic} for $c \in C$, and y_{kt} is multiplied by μ_{it}. This amounts to considering $p = 1$ in Procedure 2, where $k = i_1$ is the vertex from $G[i, p]$ not belonging to G. However, in our implementation, instead of modifying the graph by adding a new vertex, we take k to be a suitable vertex outside $G_{(\pi,\mu)}$ and so this amounts to applying Procedure 2 to the instance $(G \backslash \{k\}, H \backslash \{k\}, C)$. The final result is an inequality involving the triangle $\{i, j, k\}$ instead of the edge ij, within the same instance. This allows for a fast implementation with no need of modifying the graph and the variable set.

In this step, ij is chosen to be an edge associated to an expandable edge in the template (i.e., $ij = v(u)v(w)$ for some $uw \in XE^T$ in the configuration associated with the template), hence the vertex ℓ is fixed in this construction by the bijection v. Since the configuration satisfies the hypotheses for Procedure 2, i and j are true twins in $G_{(\pi,\mu)}$ and false twins in $H_{(\pi,\mu)}$, implying that these hypotheses remain valid in the iterative application of this procedure.

The resulting procedure is summarized in the pseudocode in Algorithm 1. When applying Procedure 2, if there is more than one vertex k providing the largest increase to the LHS, we take the first such vertex in order.

Algorithm 1 Expand the valid inequality $\pi x + \mu y \leq \pi_0$

```
for all color c ∈ XC^T do
  for all color c' ∈ C not used in π do
    if πx + μy increases when applying Procedure 1 to (π, μ) then
      Apply Procedure 1 to (π, μ) for color c with D = {c', c''}
      Replace color c'' by c in the obtained inequality
    end if
  end for
end for
for all edge ij ∈ XE^T do
  K ← {i, j}
  while there exists k ∈ V\V_(π,μ) s.t. K ∪ {k} is a clique in G and the LHS
  of πx + μy increases when applying Procedure 2 to (π, μ) do
    Take k to be the vertex providing the largest increase to the LHS, break-
    ing ties arbitrarily
    Apply Procedure 2 to (π, μ) for ij and the clique {i, j, k}
  end while
end for
```

This procedure can potentially generate cuts coming from most of the families of valid inequalities listed in Section 2 (although it is not guaranteed that cuts coming from every such families will eventually be generated). In Table 1 we summarize how these inequalities can be obtained from the templates via the application of the facet-preserving procedures. In contrast, the branch and cut procedure presented in [1] resorts to a tailored separation procedure for each family of valid inequalities. We compared both approaches within the same implementation, in order to assess the computational effectiveness of the single procedure proposed in this section.

Table 2 shows running times for a set of instances coming from a real setting and for the randomly-generated instances considered in [1]. The implementation was performed within the Cplex 12.5 environment, and the experiments were carried out on a computer with an Intel Core 2 Duo CPU, with two T8100 cores running at 2 GHz, and 2 GB of RAM memory. We have kept all Cplex parameters at their default values. This table shows the improvement achieved when the separation procedures considered in [1] are employed, and also shows that the performance of the single procedure presented in this work is quite competitive with respect to these results. The columns labeled "All templates" correspond to employing all templates mentioned before, whereas the columns labeled "T_2+T_4" correspond to considering the templates T_2 and T_4 only, which achieved the best performance. We have set a time limit of 10 minutes for these experiments.

As Table 2 shows, the families of valid inequalities (7)-(12) help reduce the running times of a branch-and-bound procedure, mainly due to the fact that their application greatly enhances the dual bound provided by the linear relaxation of the model (1)-(6). The template-based separation procedure presented in this section achieves a similar performance, both in terms of the total running time and the nodes in the enumeration tree. This suggests that the separation approach proposed in this section is effective, at least when compared with standard separation procedures. It is interesting to note that the instances coming from real scenarios are easier to solve than randomly-generated instances of smaller sizes. This is probably due to the structure that real instances usually have (e.g., the graph G is usually an interval graph), and that is not present in randomly-generated graphs.

Table 3 compares the number of cuts generated in each case. It is interesting to note that the procedure presented in this work finds a much smaller number

Base template	Applied procedures	Resulting inequality
T_1	Proc. 1 on c	(7)
T_4	Proc. 2 on jk	(8)
T_5	Proc. 1 on d	(10)
T_6	Proc. 1 on c	(11)
T_7	Proc. 1 on c and Proc. 2 on jk	(12)
T_3	Proc. 2 on ij	$\sum_{k\in K} x_{kc} \leq 1$

Table 1: Construction of the inequalities presented in Section 2 from the templates considered in Section 4 by the application of the facet-preserving procedures.

of cuts, while obtaining a similar performance. This is due to the fact that the separation procedure for the semi-diamond inequalities used in [1] generates many inequalities. A better tuning of this procedure may generate a smaller number of cuts coming from this family, thus making the difference in the number of cuts less important. Nevertheless, it is not clear whether such a better tuning may be attained without resorting to the techniques presented in this work. The iterative application of both procedures within Algorithm 1 allows to find inequalities with a potentially large set of colors D in Procedure 1 and a potentially large clique K in G in Procedure 2. In our experiments, the average cardinality of D was 1.25 (with a maximum value of $|D| = 7$) for the first group of instances, and the average cardinality of D was 2.16 (with a maximum value of $|D| = 3$) for the randomly-generated instances. Similary, the average achieved clique size in Procedure 2 was 2.22 (with a maximum value of $|K| = 6$) for the first group of instances, and this average was 3.07 (with a maximum value of $|K| = 4$) for the randomly-generated instances.

The number of matches and the number of generated cuts depend on the value of ε (recall that an inequality coming from a template is considered if it is violated by at least ε, and if $\varepsilon \leq 0$ then non-violated inequalities can be accepted). If ε takes a large negative value, then many inequalities are selected but not all of them may finally generate cuts (i.e., violated inequalities) after applying Procedures 1 and 2. On the other hand, if ε takes a large positive value then the template matching procedure identifies fewer inequalities, although in this case the running time may be smaller. In order to evaluate these observations in practice, Figure 2 shows the behavior of the template-based separation procedure for the instance 2014.02.I as a function of ε. As expected, some of the non-violated inequalities do not generate cuts when $\varepsilon \leq 0$, whereas all matched templates identified for $\varepsilon > 0$ generate cuts. Running times of the separation procedure go from 17.29 seconds (for $\varepsilon = -0.5$) to 4.61 seconds (for $\varepsilon = 0$) and, with the exception of some spikes, remains around this value for $\varepsilon > 0$.

The number of available colors for the instances in Table 2 are quite close to $\chi(G)$, and this may distort the measurements. In order to evaluate the behavior of the overall procedure when more colors are available, we report in Table 4 the running time needed to solve to optimality the real-world instances considered in this work when up to five additional colors can be used. These measurements show a slight trend towards shorter running times when additional colors are available, with the exception of the instance 2014.02.II, for which this decrease in the running times is more marked.

It is also interesting to explore the impact of the graph H on the behavior of the overall procedure. To this end, Figure 3 reports the time to optimality in seconds and the number of nodes in the enumeration tree for randomly-generated instances between 20 and 30 vertices. For each instance, the graph G is kept fixed and the graph H is randomly generated having between 0 and 95 edges. This allows to measure the impact of a growing graph H on the performance. Since instances require different running times (resp. numbers of nodes), we have normalized the measurements of each instance by dividing the total time (resp. nodes) by the measurement achieved by the instance with

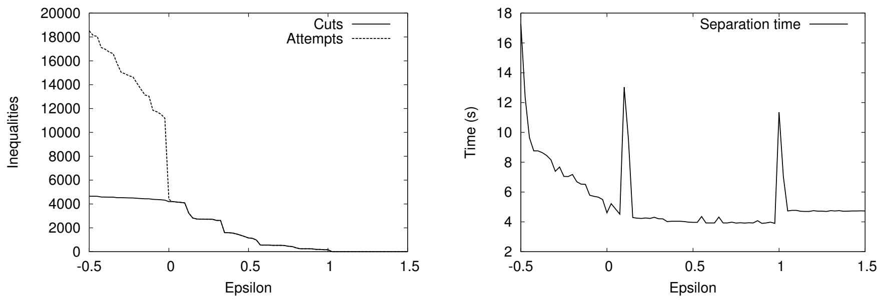

Figure 2: (a) Matched templates and generated cuts and (b) running time of the separation procedure for the instance 2014.02.I as a function of ε.

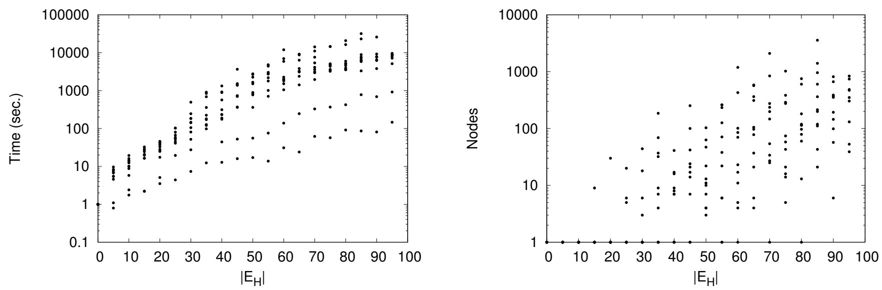

Figure 3: (a) Time to optimality and (b) nodes in the enumeration tree (vertical axis in logarithmic scale) for randomly-generated instances with H randomly generated from 0 to 95 edges, normalized so the value 1 in the vertical axis corresponds to the measurement for $|E_H| = 0$.

$|E_H| = 0$. This allows us to capture the overall trend by comparing relative values. The instances that could not be solved to optimality within 10 minutes are not shown in these figures. These experiments show that the larger the graph H, the larger the running time and number of nodes in the enumeration tree needed to achieve optimality.

5. Conclusions

From a theoretical point of view, it is interesting to provide a unified framework explaining many families of facet-inducing inequalities. The facetness proofs provided in [1] contain similar ideas that are repeated many times and that are applied with almost no variations in different proofs, so the facet-preserving procedures presented in this work allow for more elegant proofs of these results. From a practical point of view, our computational experiments show that –at least for the instances considered in this work– it is not necessary

						Cplex 12.5		Traditional cuts [1]		All templates		$T_2 + T_4$	
Instance	$\|V\|$	$\|E_G\|$	$\|E_H\|$	$\|C\|$	$\chi(G)$	Time (s)	Nodes	Time (s)	Nodes	Time (s)	Nodes	Time (s)	Nodes
2010.01.I	235	1894	124	21	21	**2.57**	1	2.59	1	2.78	1	2.6	1
2010.01.II	267	2593	299	22	22	83.98	70	12.97	1	13.28	1	**12.78**	1
2010.02.I	256	1884	118	18	18	136.87	15741	2.77	1	2.79	1	**2.67**	1
2010.02.II	279	2914	164	26	26	8.44	32	5.25	1	5.47	1	**5.08**	1
2011.01.I	265	2092	137	16	15	26.48	72	7.87	1	**6.46**	1	6.52	1
2011.01.II	255	2295	218	20	20	4.70	3	4.25	1	4.11	1	**3.96**	1
2012.01.I	182	1381	113	18	18	98.66	2122	**2.50**	1	2.64	1	2.6	1
2012.01.II	235	2220	189	23	23	44.83	1000	4.09	1	4.04	1	**3.86**	1
2012.02.I	253	1974	162	20	19	31.09	271	2.72	1	2.93	1	**2.6**	1
2012.02.II	254	2368	186	22	22	2.42	1	**2.42**	1	2.87	1	2.79	1
2014.02.I	172	1201	266	20	15	6.88% *	10799	7.00	1	5.61	1	**5.59**	1
2014.02.II	238	2294	498	20	20	29.13% *	105	459.26	21	401.71	25	**242.44**	23
rand.01	20	47	40	15	15	5.54	781	2.33	5	1.82	1	**1.79**	21
rand.02	22	60	53	16	16	18.64	1614	10.91	31	7.01	19	**3.99**	38
rand.03	24	72	56	18	18	51.42	5005	35.26	527	56.62	636	**5.41**	217
rand.04	26	83	68	19	19	46.59	1133	25.35	45	15.76	8	**3.33**	1
rand.05	28	110	77	21	21	28.90% *	11300	95.26	324	238.53	476	**22.14**	129
rand.06	30	103	96	22	22	34.88% *	6043	398.52	4913	343.66	2865	**107.21**	3011
rand.07	32	123	103	24	24	49.66% *	5788	488.8	7038	419.39	6500	**177.42**	8752
rand.08	34	141	124	25	25	81.11% *	2729	233.74	1301	329.52	1716	**90.94**	1441
rand.09	36	166	141	27	27	59.61% *	2311	6.03% *	8586	12.65% *	5800	**3.58% ***	21400
rand.10	38	173	147	28	28	108.65% *	1167	27.99% *	5827	26.20% *	3800	**18.55% ***	15140

Table 2: Time to optimality and nodes in the enumeration tree for Cplex as a black-box solver (columns labeled “Cplex 12.5”), for the cut-and-branch with all the separation procedures presented in [1] (columns labeled “Traditional cuts [1]”), for a cut-and-branch using the cut-generating procedure presented in this section with all templates (columns labeled “All templates”), and for a cut-and-branch using the templates T_2 and T_4 only (columns labeled “$T_2 + T_4$”), respectively. For the instances marked with “*”, the time limit of 10 minutes was attained, and in these cases the achieved optimality gap is reported.

Instance	Traditional Cuts	All templates Attempts	Cuts	$T_2 + T_4$ Attempts	Cuts
2010.01.I	0	0	0	0	0
2010.01.II	6020	77	7	63	7
2010.02.I	4200	61	9	47	9
2010.02.II	3652	53	5	41	5
2011.01.I	5590	95	17	65	17
2011.01.II	2440	16	4	12	4
2012.01.I	5472	76	16	68	16
2012.01.II	4531	93	43	33	7
2012.02.I	4736	42	12	36	12
2012.02.II	0	0	0	0	0
2014.02.I	22538	341	42	309	42
2014.02.II	278312	12087	4429	3968	330
rand.01	2125	599	359	287	47
rand.02	8023	4201	3565	697	61
rand.03	14247	12467	11347	1187	67
rand.04	17613	7238	6422	902	86
rand.05	48733	33949	32409	1669	129
rand.06	27244	9572	8420	1268	116
rand.07	34685	10973	9737	1373	137
rand.08	76319	21919	20431	1711	223
rand.09	124559	32317	30625	1981	289
rand.10	103798	26299	24535	2011	247

Table 3: Number of cuts found by the individual separation procedures for each family of valid inequalities (column labeled "Traditional cuts [1]"), and by the procedure presented in this section (remaining columns), respectively. For the template-based cuts, the number of matched templates is reported in the column "Attempts", and the number of violated inequalities is reported in the column "Cuts".

		Time (sec.)					
Instance	c	$\|C\| = c$	$\|C\| = c + 1$	$\|C\| = c + 2$	$\|C\| = c + 3$	$\|C\| = c + 4$	$\|C\| = c + 5$
2010.01.I	21	2.78	2.80	2.82	3.57	2.20	3.17
2010.01.II	22	13.28	6.79	8.17	9.39	6.94	8.21
2010.02.I	18	2.79	3.12	2.79	2.35	2.03	2.25
2010.02.II	26	5.47	5.07	4.49	5.49	4.90	5.04
2011.01.I	15	6.46	5.66	4.11	3.85	3.26	2.78
2011.01.II	20	4.11	4.13	3.38	3.57	3.22	2.82
2012.01.I	18	2.64	3.33	1.98	2.50	2.04	1.86
2012.01.II	23	4.04	2.96	3.01	3.64	3.22	3.13
2012.02.I	19	2.93	2.34	2.87	4.13	3.89	2.34
2012.02.II	22	2.87	2.54	2.62	2.63	3.56	3.02
2014.02.I	15	5.61	3.51	4.01	3.11	5.00	4.54
2014.02.II	20	401.71	193.56	239.91	93.46	59.63	110.16

Table 4: Time to optimality for a cut-and-branch using the cut-generating procedure presented in this section with all templates, for increasing numbers of colors. The column "c" contains the number of colors used in Table 2, and the subsequent columns report the time to optimality when considering $c, \ldots, c + 5$ colors, respectively.

to resort to a particular separation procedure for each family of valid inequalities, and that a single cut-generating procedure based on the ideas presented in this work allows to obtain similar computational results instead. This is interesting when implementing a branch and cut procedure, since only one separation procedure must be implemented, and the templates considered in such a procedure can be easily configured.

Not all the inequalities presented in [1] can be explained in terms of simple templates and the procedures presented in Section 3. For example, we cannot apply Procedure 1 to the template T_7, namely

$$y_{ij} \leq 2 - (x_{ic} + x_{kc} + x_{\ell c}) \tag{23}$$

for $i, j, k, \ell \in V$ such that $jk, j\ell, k\ell \in E_G$ and $ij \in E_H$, and for $c \in C$, in order to obtain the (indeed valid) bounding inequality

$$y_{ij} \leq 2 - \sum_{d \in D} (x_{id} + x_{kd} + x_{\ell d}) \tag{24}$$

for $D \subseteq C$, since (23) does not satisfy the hypothesis (ii) in Procedure 1 (indeed, $\pi_{kc} > 0$). Due to this fact, we take $XC^{T_7} = \emptyset$, in order to prevent the application of Procedure 1 to this template. Nevertheless, (24) involves the ideas present in the families (7)-(11) by considering a subset D of colors instead of a single color c, hinting that it might be possible to state a generalized version of Procedure 1 not having the hypothesis (ii).

The ideas presented in Procedure 2 can be applied to the standard formulation of the classical vertex coloring formulation by ignoring the y-variables, as Corollary 3 illustrates. For example, the application of Corollary 3 to the constraint $x_{ic} + x_{jc} \leq 1$, for $ij \in E_G$ and $c \in C$, yields the *clique inequality* $\sum_{k \in K} x_{kc} \leq 1$, which is facet-inducing if $|C| > \chi(G)$ and K is a maximal clique in G. However, the hypothesis (b) of Theorem 4 asks for edges in H, which exceeds the setting of the classical vertex coloring polytope, so this theorem cannot be directly applied in order to show that the clique inequalities define facets when K is a maximal clique in G. It would be interesting to explore whether Theorem 4 can be generalized in order to cover this case as well.

Acknowledgment. The authors would like to express their gratitude towards the anonymous reviewer for the thorough and detailed comments, which helped to greatly improve this manuscript.

References

[1] Braga, M., D. Delle Donne, R. Linfati, and J. Marenco, *The maximum-impact coloring polytope*, Intl. Trans. in Op. Res. **24** (2017), 303–324.

[2] Braga M. and J. Marenco, *Facet-generating procedures for the maximum-impact coloring polytope*. Electronic Notes in Theoretical Computer Science 346 (2019) 199–208.

[3] Burke, E., J. Mareček, A. Parkes, and H. Rudová, *A branch-and-cut procedure for the Udine course timetabling problem*, Annals of Operations Research **194** (2012), 71–87.

[4] Daskalaki, S., T. Birbas, and E. Housos, *An integer programming formulation for a case study in university timetabling*, European Journal of Operational Research **153** (2004), 117–135.

[5] De Werra, D., *An introduction to timetabling*, European Journal of Operational Research **19-2** (1985), 151–162.

[6] Garey, M., and D. Johnson, "Computers and intractability: A guide to the theory of NP-completeness", W. H. Freeman, 1979.

[7] Lach, G., and M. Lübbecke, *Curriculum based course timetabling: new solutions to Udine benchmark instances,* Annals of Operations Research, vol. 194, pp. 255–272, 2012.

[8] Miranda, J., *eClasSkeduler: A course scheduling system for the Executive Education Unit at the Universidad de Chile*, Interfaces **40-3** (2010), 196–207.

[9] Mooney, E., R. Rardin, and W. Parmenter, *Large-scale classroom scheduling*, IIEE transactions **28-5** (1996), 369–378.

[10] Waterer, H., *A zero-one integer programming model for room assignment at the University of Auckland*, Proceedings of the 1995 ORSNZ Conference, 1995.