The maximum 2D subarray polytope: facet-inducing inequalities and polyhedral computations

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Abstract

Given a matrix with real-valued entries, the *maximum 2D subarray problem* consists in finding a rectangular submatrix with consecutive rows and columns maximizing the sum of its entries. In this work we start a polyhedral study of an integer programming formulation for this problem.We thus define the *2D subarray polytope*, explore conditions ensuring the validity of linear inequalities, and provide several families of facet-inducing inequalities. We also report computational experiments assessing the reduction of the dual bound for the linear relaxation achieved by these families of inequalities.

Keywords: maximum subarray problem, integer programming, facets

1. Introduction

In this work we are interested in the maximum 2D subarray problem, which consists in finding a submatrix (with consecutive rows and columns) of a real-valued matrix maximizing the sum of its entries. This problem arises in the column generation phase of an integer-programming-based procedure for solving the rectilinear picture compression problem. In this section, we present the latter problem in order to motivate this work.

The *rectilinear picture compression problem* (RPC) consists in covering all entries with value 1 of a binary matrix $M \in \{0,1\}^{m\times n}$ with a minimum number of submatrices having contiguous rows and columns formed of entries with value 1. We call a *rectangle* the set of elements of any such submatrix. Figure 1(i) shows an example instance of this problem, which can be covered with a minimum number of three rectangles, namely the rectangles depicted in Figure 1(ii)-(iv). The rectangles in a solution need not be disjoint nor maximal.

Although the initial motivation for exploring RPC comes from the compression of monochromatic images (in particular, monochromatic images coming from the union of a few rectangles), this problem also has applications in the synthesis of DNA arrays [1] and in the processing of *access control lists (ACLs)* in network routers [2].

The earliest reference to RPC seems to be due to Masek [3]. In this work, the author showed that RPC is NP-hard; Berman and DasGupta later proved

1	1	0	1
1	1	0	1
1	1	1	1
0	1	1	1

(*i*)

1	1		
1	1		
1	1		

(*ii*)

	1	1	1
	1	1	1

(*iii*)

			1
			1
			1

(*iv*)

Figure 1: Subfigure (i) shows an example of a 4×4 instance, and subfigures $(ii) - (iv)$ show the rectangles in a three-rectangle solution for this instance.

it to be MaxSNP-hard [4]. The best known polynomial-time approximation guarantee is $O(\sqrt{\log k})$, where k is the number of entries with value 1 in the input matrix [5]. A slightly more general version of the problem has also been studied under a polyhedral approach in [6, 7]. In the former, the authors analyze new lower bounds on the optimal cover size based on the fractional solution of the linear programming relaxation of the proposed formulation. The latter discusses two integer programming models for the rectangle cover of a convex polygon.

Given a binary matrix $M \in \{0,1\}^{m \times n}$, we define formally a *rectangle* in M as the set $\{(k,\ell) : i_1 \leq k \leq i_2 \text{ and } j_1 \leq \ell \leq j_2\}$ for some indices $i_1, i_2 \in \{1,\ldots,m\}$ and $j_1, j_2 \in \{1,\ldots,n\}$. We define $R(M)$ to be the set of rectangles in M that contain only entries with value 1 in M, namely $R(M) = \{r : r \text{ is a rectangle in } M \text{ and } M_{ij} = 1 \text{ for every } (i,j) \in r\}$. For each $r \in R(M)$, we introduce a binary variable x_r specifying whether the rectangle r is chosen in the cover or not. For every entry (i,j) with $M_{ij} = 1$, at least one rectangle containing (i,j) has to be selected, and we seek to minimize the number of selected rectangles. This leads to the following integer programming formulation for RPC.

$$\min \quad \sum_{r \in R(M)} x_r \tag{1}$$

$$M_{ij} \leq \sum_{r \in R(M):(i,j) \in r} x_r \qquad i = 1,\ldots,m,\ j = 1,\ldots,n, \tag{2}$$

$$x_r \in \{0,1\} \qquad r \in R(M). \tag{3}$$

The number of rectangles in $R(M)$ is polynomial in the size of M, namely $|R(M)| \in O(n^2 m^2)$. However, for a medium to large-sized input matrix, this formulation quickly leads to an impractical large number of variables. In this setting, a natural solution approach is column generation, i.e., the dynamic generation of rectangle variables for the linear relaxation of the formulation when strong duality is violated. Column generation consists in this case in finding a (weighted) rectangle of negative reduced cost. We seek a 2-dimensional array of maximum weight within M, where the weights are given by the dual variables associated with constraints (2). This corresponds to solving the maximum 2D subarray problem. The study of the polytope associated to this problem is the main focus of our work.

We organize the remainder of this paper as follows. Section 2 introduces the maximum 2D subarray problem, whereas Section 3 presents an integer programming formulation for this problem and defines the associated polytope. Section 4 contains the main results of this work, including facet-inducing families of inequalities for this polytope together with technical lemmas. Section 5 reports computational experiments performed in order to evaluate the reduction of the dual bound provided by the linear relaxation, when the inequalities presented in Section 5 are added to the model. Finally, Section 6 closes the paper with concluding remarks. A preliminary version of the results in Section 4 appeared without proofs in the conference paper [8].

2. The maximum 2D subarray problem

This section introduces formally the maximum 2D subarray problem. Given a d-dimensional real-valued array $A \in \mathbb{R}^{n_1 \times \cdots \times n_d}$, $d \geq 1$, the *maximum subarray problem* consists in finding a contiguous and axis-parallel section of A with maximum sum. We are interested in the case $d = 2$, which corresponds to a 2-dimensional array $A \in \mathbb{R}^{m \times n}$, and asks for row indices $i_1, i_2 \in \{1, \ldots, m\}$, $i_1 \leq i_2$, and column indices $j_1, j_2 \in \{1, \ldots, n\}$, $j_1 \leq j_2$, such that $\sum_{i=i_1}^{i_2} \sum_{j=j_1}^{j_2} A_{ij}$ is maximum. This problem is called the *maximum 2D subarray problem*. In our setting A contains real-valued entries, coming from the dual variables associated with constraints (2) in the model for RPC.

In this work we start such an issue, by exploring the polytope associated with a natural integer programming formulation of this problem. The final objective of such an undertaking is to identify strong families of valid inequalities for the polytope. The results presented in the following sections do not depend on the entries of A since the polytope definition does not involve the objective function being optimized. Nevertheless, we need to take the entries of A into account when designing separation procedures for families of valid inequalities for this polytope.

The associated polytope is a two-dimensional version of the *full interval vectors polytope*, i.e., the convex hull of vectors in $\{0, 1\}^n$ having consecutive ones. This polytope has been studied in [9] and the results therein have inspired some of the results in the current work.

3. The 2D subarray polytope

Consider a real-valued matrix $A \in \mathbb{R}^{m \times n}$ with m rows and n columns. Denote by $R = \{1, \ldots, m\}$ the set of row indexes, and by $C = \{1, \ldots, n\}$ the set of column indexes. We also define $P = R \times C$ to be the set of *entries* of A (also called *pixels* in this context). For $(i, j) \in P$, we introduce the binary variable x_{ij}, which takes value 1 if and only if the pixel (i, j) belongs to the solution rectangle.

The *rectangular hull* of a nonempty set $S \subseteq P$ of pixels, denoted by $\square(S)$, is the smallest rectangle including all the pixels in S, i.e., $\square(S) = \{(k,\ell) : \min_{(i,j)\in S} i \le k \le \max_{(i,j)\in S} i \text{ and } \min_{(i,j)\in S} j \le \ell \le \max_{(i,j)\in S} j\}$. If $S = \{p, p'\}$ with $p = (i,j)$ and $p' = (i',j')$, then we denote $\square(S)$ by $\square(p,p')$ and by $\square(i,j,i',j')$. If $S = \{p\}$, then we denote $\square(S)$ by $\square(p)$. We also define $\square(\emptyset) = \emptyset$.

For $S \subseteq P$, we define $\blacksquare(S)$ to be the feasible solution $x \in \{0,1\}^{mn}$ having $x_{ij} = 1$ if and only if $(i,j) \in \square(S)$. The solutions $\blacksquare(i,j,i',j')$, $\blacksquare(p,p')$, and $\blacksquare(p)$ for $p = (i,j)$ and $p' = (i',j')$ are defined similarly.

Definition 1. *For $m \in \mathbb{Z}_+$ and $n \in \mathbb{Z}_+$, we define* $P^{\square}_{m,n} = \mathrm{conv}\big(\{\mathbf{0}\} \cup \{\blacksquare(i,j,i',j') : 1 \le i \le i' \le m \text{ and } 1 \le j \le j' \le n\}\big)$.

We now give a formulation for the 2D maximum subarray problem as an optimization problem over $P^{\square}_{m,n}$. For $(i,j) \in P$, the value $A_{ij} \in \mathbb{R}$ is the benefit associated with picking the pixel (i,j). In this setting, the 2D maximum subarray problem can be formulated as follows.

$$\max \sum_{(i,j)\in P} A_{ij} x_{ij} \tag{4}$$

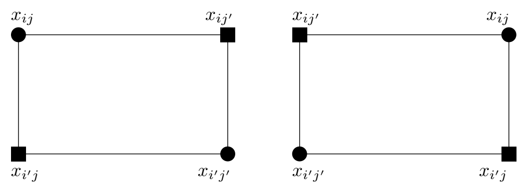
$$\begin{aligned}
x_{ij} + x_{ij'} &\le x_{i(j-1)} + 1 && (i,j) \in P,\ j > 2,\ j' \le j-2, && (5)\\
x_{ij} + x_{i'j} &\le x_{(i-1)j} + 1 && (i,j) \in P,\ i > 2,\ i' \le i-2, && (6)\\
x_{ij} + x_{i'j'} &\le x_{ij'} + 1 && (i,j),(i'j') \in P,\ i < i',\ j \ne j', && (7)\\
x_{ij} + x_{i'j'} &\le x_{i'j} + 1 && (i,j),(i'j') \in P,\ i < i',\ j \ne j', && (8)\\
x_{ij} &\in \{0,1\} && (i,j) \in P. && (9)
\end{aligned}$$

Constraints (5) (resp. (6)) force that an element of a row (resp. a column) between two columns (resp. two rows) in the solution must belong to it. Constraints (7) and (8) ensure that if pixels (i,j) and (i',j'), with $i < i'$, are part of the solution rectangle, then pixels (i,j') and (i',j) are contained in the rectangle as well. These two families of constraints are illustrated in Fig. 2. Constraints (7) and (8) can be replaced by the weaker constraint $x_{ij} + x_{i'j'} \le \frac{x_{ij'} + x_{i'j}}{2} + 1$, which directly follows from them. The results in Section 4 imply that constraints (5)-(8) induce facets of $P^{\square}_{m,n}$.

The convex hull of feasible solutions to (5)-(9) coincides with $P^{\square}_{m,n}$. Note that we allow the null solution to be feasible, namely $\mathbf{0} \in P^{\square}_{m,n}$. This implies the following result.

Proposition 1. *$P^{\square}_{m,n}$ is full-dimensional.*

Proof. For each pixel $(i,j) \in P$, the solution $\blacksquare(i,j)$ belongs to $P^{\square}_{m,n}$. This fact, together with $\mathbf{0} \in P^{\square}_{m,n}$, implies the result. $\square$



(a) Case where $j' > j$ (b) Case where $j' < j$

Figure 2: Constraints (7) and (8) of the formulation for the maximum 2D subarray problem. The rectangles represent $\square(i,j,i',j')$. All the x-variables related to the rectangles in cases (a) and (b) must be set to 1 if $x_{ij} = x_{i'j'} = 1$.

4. Facets of the 2D subarray polytope

Let $\pi \in \mathbb{Z}^{mn}$. For $c \in \mathbb{Z}$, we define $I^\pi_c = \{(i,j) \in P : \pi_{ij} = c\}$ to be the set of pixels with coefficient c in π. We also define $I^\pi_{>0}$ to be the set of pixels with positive coefficient in π, and $I^\pi_{<0}$ to be the set of pixels with negative coefficient in π. Finally, we define $I^\pi_{\neq 0} = I^\pi_{>0} \cup I^\pi_{<0}$. These definitions allow us to provide a general characterization of valid inequalities for $P^\square_{m,n}$.

Theorem 1. *Let $\pi \in \mathbb{Z}^{mn}$ and $\pi_0 \in \mathbb{Z}$. The inequality $\pi x \leq \pi_0$ is valid for $P^\square_{m,n}$ if and only if*

$$\sum_{(i,j)\in S} \pi_{ij} + \sum_{(i,j)\in \square(S)\cap I^\pi_{<0}} \pi_{ij} \;\leq\; \pi_0 \tag{10}$$

for every $S \subseteq I^\pi_{>0}$.

Proof. ($\Rightarrow$) Let $\pi x \leq \pi_0$ be a valid inequality for $P^\square_{m,n}$ and $\bar{S} \subseteq I^\pi_{>0}$ induce a feasible solution $\blacksquare(\bar{S})$. We have

$$\begin{aligned} \pi_0 \;\geq\; \pi\blacksquare(\bar{S}) \quad &= \sum_{(i,j)\in\square(\bar{S})\cap I^\pi_{>0}} \pi_{ij}\bar{x}_{ij} + \sum_{(i,j)\in\square(\bar{S})\cap I^\pi_{<0}} \pi_{ij}\bar{x}_{ij} \\ &= \sum_{(i,j)\in\square(\bar{S})\cap I^\pi_{>0}} \pi_{ij} + \sum_{(i,j)\in\square(\bar{S})\cap I^\pi_{<0}} \pi_{ij}, \end{aligned}$$

where this last equality follows from the fact that the $\bar{x}$-components related to $\square(\bar{S})$ are equal to 1. Because $\sum_{(i,j)\in\square(\bar{S})\cap I^\pi_{>0}} \pi_{ij} \geq \sum_{(i,j)\in S} \pi_{ij}$ for every subset $S \subseteq \square(\bar{S}) \cap I^\pi_{>0}$, including when $S = \bar{S}$, the result follows.

($\Leftarrow$) Assume that $\sum_{(i,j)\in S} \pi_{ij} + \sum_{(i,j)\in\square(S)\cap I^\pi_{<0}} \pi_{ij} \leq \pi_0$ for every subset $S \subseteq I^\pi_{>0}$. Let $\bar{x} \in \{0,1\}^{mn}$ represent an arbitrary feasible solution for $P^\square_{m,n}$, and let $A \subseteq P$ be the rectangle represented by $\bar{x}$. Define $\bar{S} = A \cap I^\pi_{>0}$. In this case,

$$\begin{aligned}\pi\bar{x} \ = \ \pi\blacksquare(A) \ &= \sum_{(i,j)\in A\cap I^{\pi}_{>0}} \pi_{ij}\bar{x}_{ij} + \sum_{(i,j)\in A\cap I^{\pi}_{<0}} \pi_{ij}\bar{x}_{ij} \\ &= \sum_{(i,j)\in A\cap I^{\pi}_{>0}} \pi_{ij} + \sum_{(i,j)\in A\cap I^{\pi}_{<0}} \pi_{ij} \\ &\leq \sum_{(i,j)\in \bar{S}} \pi_{ij} + \sum_{(i,j)\in \square(\bar{S})\cap I^{\pi}_{<0}} \pi_{ij} \ \leq \ \pi_0,\end{aligned}$$

where the before-to-last inequality follows from the fact that $\square(\bar{S}) \subseteq A$ and the coefficients in the second summation are negative, and the last one follows from the assumption. This concludes the proof. □

Although difficult to check in practice, the condition ensuring validity in Theorem 1 will be useful in the next sections.

4.1. Facet-inducing inequalities with coefficients in $\{-1, 0, 1\}$

We first explore valid inequalities with coefficients in $\{-1, 0, 1\}$. In this case, Theorem 1 implies the following characterization for validity.

Corollary 1. *Let* $\pi \in \{-1, 0, 1\}^{mn}$*. The inequality* $\pi x \leq 1$ *is valid for* $P^{\square}_{m,n}$ *if and only if* $|\square(S) \cap I^{\pi}_{-1}| \geq |S| - 1$ *for every* $S \subseteq I^{\pi}_1$*.*

We say that a pixel $(k, \ell) \in I^{\pi}_0$ is *reachable in* π from the pixel $(i, j) \in P$ if $\square(i, j, k, \ell)\backslash\{(i, j)\} \subseteq I^{\pi}_0$, i.e., all pixels in the rectangular hull $\square(i, j, k, \ell)$ have coefficient 0 in π, with the exception of (i, j). For $B \subseteq P$, we define $\pi(B) := \sum_{(i,j)\in B} \pi_{ij}$ and $\lambda(B) := \sum_{(i,j)\in B} \lambda_{ij}$. Finally, we define $F^{\pi} := \{x \in P^{\square}_{m,n} : \pi x = 1\}$.

Lemma 1. *Let* $\pi x \leq 1$ *be a valid inequality for* $P^{\square}_{m,n}$*. Assume* $\lambda x = \lambda_0$ *for every* $x \in F^{\pi}$*. If* $p \in I^{\pi}_1$ *and* $q \in I^{\pi}_0$ *is reachable from* p*, then* $\lambda_q = 0$*.*

Proof. Assume w.l.o.g. $p = (i_1, j_1)$ and $q = (i_2, j_2)$, with $i_1 \leq i_2$ and $j_1 \leq j_2$. If $i_1 = i_2$, then $j_1 < j_2$ since $p \neq q$. Consider the feasible solutions $\bar{x}^1 := \blacksquare(i_1, j_1, i_1, j_2 - 1)$ and $\bar{x}^2 := \blacksquare(i_1, j_1, i_1, j_2)$. Since $\pi_p = 1$ and $\pi_{i_1 j} = 0$ for $j = j_1 + 1, \ldots, j_2$, we have $\bar{x}^1, \bar{x}^2 \in F^{\pi}$, hence $\lambda\bar{x}^1 = \lambda\bar{x}^2$. Since $\bar{x}^1$ and $\bar{x}^2$ only differ in the variable x_q, the conclusion $\lambda_q = 0$ follows.

A similar analysis settles the case $j_1 = j_2$ (and $i_1 < i_2$), so assume $i_1 < i_2$ and $j_1 < j_2$. Consider now the feasible solutions $\bar{x}^1 := \blacksquare(i_1, j_1, i_2, j_2 - 1)$ and $\bar{x}^2 := \blacksquare(i_1, j_1, i_2 - 1, j_2 - 1)$. Also define $B := \square(i_2, j_1, i_2, j_2 - 1)$ and $R := \square(i_1, j_2, i_2 - 1, j_2)$ (see Figure 3). Since $\pi_r = 0$ for every $r \in \square(p, q)\backslash\{p\}$, then $\bar{x}^1 \in F^{\pi}$ and $\bar{x}^2 \in F^{\pi}$. This implies $\lambda\bar{x}^1 = \lambda\bar{x}^2$, hence $\lambda(B) = 0$. Consider now the solution $\bar{x}^3 := \blacksquare(i_1, j_1, i_2 - 1, j_2)$. Again, $\bar{x}^3 \in F^{\pi}$, hence $\lambda\bar{x}^2 = \lambda\bar{x}^3$, implying $\lambda(R) = 0$. Finally, let $\bar{x}^4 := \blacksquare(p, q)$. Again, $\bar{x}^4 \in F^{\pi}$, so $\lambda\bar{x}^2 = \lambda\bar{x}^4$, and this implies $\lambda(B) + \lambda(R) + \lambda_q = 0$. Since $\lambda(B) = \lambda(R) = 0$, the conclusion follows. □

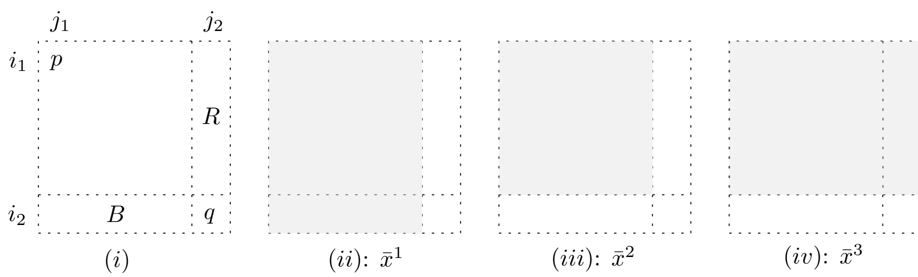


Figure 3: Constructions for the proof of Lemma 1.

Lemma 1 will be the basis for many facetness results as, e.g., the following theorem. From now on, when we refer to a reachable pixel q in the expression $\pi x = 1$ and $\lambda x = \lambda_0$ for every $x \in F^\pi$, we assume $\lambda_q = 0$.

Theorem 2. *Let $\pi x \leq 1$ be a valid inequality with $\pi \in \{-1, 0, 1\}^{mn}$. If (a) every pixel in I_0^π is reachable from some pixel in I_1^π and (b) for every $p \in I_{-1}^\pi$ there exist $q, q' \in I_1^\pi$ such that $\square(q, q') \cap I_1^\pi = \{q, q'\}$ and $\square(q, q') \cap I_{-1}^\pi = \{p\}$, then $\pi x \leq 1$ defines a facet of $P_{m,n}^\square$.*

Proof. Let (λ, λ_0) such that $\lambda x = \lambda_0$ for every $x \in F^\pi$. The following claims settle this proof.

- Let $p_1, p_2 \in I_1^\pi$. The solutions $\blacksquare(p_1)$ and $\blacksquare(p_2)$ belong to F, implying $\lambda_{p_1} = \lambda_{p_2}$. This implies that there exists $\alpha \in \mathbb{R}$ such that $\lambda_p = \alpha = \lambda_0$ for every $p \in I_1^\pi$.
- Let $q \in I_0^\pi$. By the hypothesis (a), there exists $p \in I_1^\pi$ such that q is reachable from p. Lemma 1 implies $\lambda_q = 0$.
- Let $p \in I_{-1}^\pi$. By the hypothesis (b), there exist $q, q' \in I_1^\pi$ such that $\square(q, q') \cap I_1^\pi = \{q, q'\}$ and $\square(q, q') \cap I_{-1}^\pi = \{p\}$. Consider the solutions $\bar{x}^1 := \blacksquare(q)$ and $\bar{x}^2 := \blacksquare(q, q')$. Since $\bar{x}^1, \bar{x}^2 \in F$, we have $\lambda \bar{x}^1 = \lambda \bar{x}^2$. Together with $\lambda_r = 0$ for every $r \in \square(q, q') \setminus \{p, q, q'\}$, this implies $\lambda_p + \lambda_{q'} = 0$, hence $\lambda_p = -\alpha$.

By combining these claims we get $\lambda = \alpha \pi$, so the result follows. $\square$

Theorem 2 allows us to derive several families of facet-inducing inequalities $\pi x \leq 1$ for $P_{m,n}^\square$ with coefficients in $\{-1, 0, 1\}$ (see Figure 4 for an example), and is the starting point for the subsequent theorems. It is important to note that Theorem 2 does not characterize all facet-inducing inequalities with coefficients in $\{-1, 0, 1\}$. Indeed, some of the following facet-inducing inequalities do not stem from this result directly.

Theorem 3. *Let $\pi \in \{-1, 0, 1\}^{mn}$. If $I_1^\pi = \{(i_1, j_1), (i_2, j_2)\}$ and $I_{-1}^\pi = \{(k, \ell)\} \subseteq \square(i_1, j_1, i_2, j_2)$, then $\pi x \leq 1$ is facet-inducing for $P_{m,n}^\square$.*

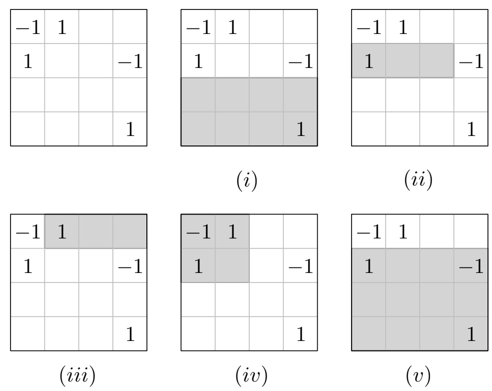


Figure 4: A valid inequality $\pi x \leq 1$ that verifies the hypotheses of Theorem 2. The inequality is represented by specifying the nonzero coefficients of π in the corresponding pixels of the input matrix. The gray rectangles in subfigures $(i)-(iii)$ show that Condition (a) of Theorem 2 is verified, while the gray rectangles in (iv) and (v) show that Condition (b) is also met.

Proof. Assume w.l.o.g. $i_1 \leq i_2$ and $j_1 \leq j_2$. If $i_1 < k < i_2$ and $j_1 < \ell < j_2$, then all pixels in I_0^π are reachable from either (i_1, j_1) or (i_2, j_2), and the result follows from Theorem 2.

So assume w.l.o.g. $\ell = j_2$ (hence $k < i_2$). Let (λ, λ_0) such that $\lambda x = \lambda_0$ for every $x \in F^\pi$. We consider the following cases (see Figure 5).

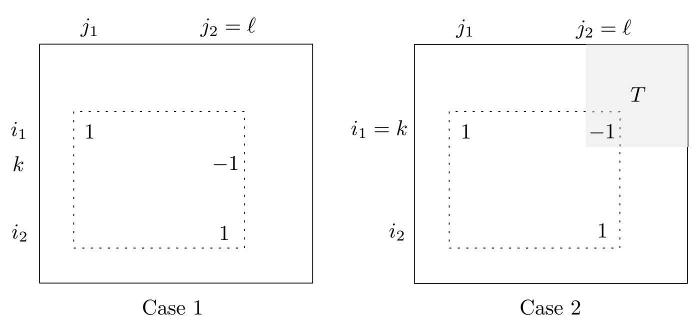


Figure 5: Cases in the proof of Theorem 3.

Case 1: $k > i_1$. All pixels in $I_0^\pi \backslash \square(k, j_2 + 1, k, n)$ are reachable from either (i_1, j_1) or (i_2, j_2), so Lemma 1 implies $\lambda_r = 0$ for every $r \in I_0^\pi \backslash \square(k, j_2 + 1, k, n)$.

For $t = \ell, \ldots, n-1$, consider the solutions $\bar{x}^1 := \blacksquare(i_1, j_1, i_2, t)$ and $\bar{x}^2 := \blacksquare(i_1, j_1, i_2, t+1)$. Both solutions belong to F^π, hence $\lambda \bar{x}^1 = \lambda \bar{x}^2$, implying

$$\sum_{i=i_1}^{i_2} \lambda_{i,t+1} = 0.$$

But $\lambda_{i,t+1} = 0$ for $i \neq k$, hence $\lambda_{k,t+1} = 0$. This implies $\lambda_r = 0$ for every $r \in I_0^\pi$.

Finally, the solutions $\blacksquare(i_1, j_1)$, $\blacksquare(i_2, j_2)$, and $\blacksquare(i_1, j_1, i_2, j_2)$ show $\lambda_{i_1,j_1} = \lambda_{i_2,j_2} = -\lambda_{k\ell}$, allowing us to conclude the result.

Case 2: $\boldsymbol{k = i_1}$. Define $T := \square(1, j_2, k, n)$. All pixels in $I_0^\pi \backslash T$ are reachable from either (i_1, j_1) or (i_2, j_2), so Lemma 1 implies $\lambda_r = 0$ for every $r \in I_0^\pi \backslash T$. We can also show $\lambda_{i_1,j_1} = \lambda_{i_2,j_2} = -\lambda_{k\ell}$ as in the previous case, so we are left to prove $\lambda_r = 0$ for every $r \in T\backslash\{(k, \ell)\}$.

For $(i, j) \in T$, define $\bar{x}^{ij} := \blacksquare(i, j_1, i_2, j)$. The solutions $\{\bar{x}^{ij}\}_{(i,j)\in T}$ belong to F^π. If we order these solutions in descending order of the first coordinate and ascending order of the second coordinate, then their projection onto the variables associated with pixels in T is a diagonal matrix with ones in the diagonal, so the points $\{\bar{x}^{ij}\}_{(i,j)\in T}$ are affinely independent. Since $\lambda_r = 0$ for every $r \in I_0^\pi \backslash T$, the existence of these solutions implies $\lambda_r = 0$ for every $r \in T\backslash\{(k, \ell)\}$.

Therefore, there exists $\alpha \in \mathbb{R}$ such that $\lambda = \alpha\pi$, and the result follows. $\square$

The following technical lemma will be useful in the remainder of this section. If $A \subseteq P$ is a subset of pixels and $\pi \in \mathbb{R}^{mn}$, we define π_A to be the projection of π onto the variables associated with the pixels in A. Similarly, if $x \in P^\square_{m,n}$, we define x_A to be the projection of x onto the variables associated with the pixels in A.

Lemma 2. *Let $\pi x \leq \pi_0$ be a valid inequality for $P^\square_{m,n}$, and let $k \in \{1, \ldots, m\}$. Define $T := \square(1, 1, k, n)$ and $B := \square(k, 1, m, n)$, and assume $\pi_{k\ell} \neq 0$ for some $(k, \ell) \in T \cap B$. If $\pi_T x_T \leq \pi_0$ induces a facet of $P^\square_{k,n}$ and $\pi_B x_B \leq \pi_0$ induces a facet of $P^\square_{m-k+1,n}$, then $\pi x \leq \pi_0$ induces a facet of $P^\square_{m,n}$.*

Proof. Let $\bar{x}^1, \ldots, \bar{x}^{|T|}$ be affinely independent points with $\pi_T \bar{x}^t = \pi_0$ for $t = 1, \ldots, |T|$, and let $\bar{y}^1, \ldots, \bar{y}^{|B|}$ be affinely independent points with $\pi_B \bar{x}^t = \pi_0$ for $t = 1, \ldots, |B|$. For $t = 1, \ldots, |T|$, we define $\hat{x}^t \in P^\square_{m,n}$ as

$$\hat{x}^t_{ij} = \begin{cases} \bar{x}^t_{ij} & \text{if } (i,j) \in T, \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Similarly, for $t = 1, \ldots, |B|$, we define $\hat{y}^t \in P^\square_{m,n}$ as

$$\hat{y}^t_{ij} = \begin{cases} \bar{y}^t_{ij} & \text{if } (i,j) \in B, \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. The points $\{\hat{x}^t\}_{t=1}^{|T|}$ are affinely independent and satisfy $\pi x \leq \pi_0$ with equality, and the same holds for $\{\hat{y}^t\}_{t=1}^{|B|}$. Let $U := \{\hat{x}^t\}_{t=1}^{|T|} \cup \{\hat{y}^t\}_{t=1}^{|B|}$.

We claim that U containts mn affinely independent points. To this end, let (γ, γ_0) such that $\gamma x = \gamma_0$ for every $x \in U$. We have $\gamma_T \bar{x}^t = \gamma_0$ for $t = 1, \ldots, |T|$ and $\gamma_B \bar{y}^t = \gamma_0$ for $t = 1, \ldots, |B|$. Since $\{\bar{x}^t\}_{t=1}^{|T|}$ has dimension $|T|-1$, then there exists $\alpha \in \mathbb{R}$ such that $\gamma_T = \alpha\pi_T$. Similarly, the fact that $\{\bar{y}^t\}_{t=1}^{|B|}$ has dimension $|B| - 1$ implies that there exists $\beta \in \mathbb{R}$ such that $\gamma_B = \beta\pi_B$. The hypothesis ensures that there exists $(k, \ell) \in T \cap B$ with $\pi_{k\ell} \neq 0$, so $\alpha\pi_{k\ell} = \gamma_{k\ell} = \beta\pi_{k\ell}$, implying $\alpha = \beta$. Therefore, $\gamma = \alpha\pi$ and then U has dimension $mn - 1$. Thus, $\pi x \leq \pi_0$ induces a facet of $P^\square_{m,n}$. $\square$

Theorem 4. *Let $\pi \in \{-1,0,1\}^{mn}$ with $|I_1^\pi| = |I_{-1}^\pi| + 1$. If $I_1^\pi = \{(i_t, j_t)\}_{t=1}^k$ with $i_t \leq i_{t+1}$ and $j_t \leq j_{t+1}$ for $t = 1, \dots, k-1$, and $\square(i_t, j_t, i_{t+1}, j_{t+1})$ contains exactly one pixel from I_{-1}^π for $t = 1, \dots, k-1$, then $\pi x \leq 1$ is facet-inducing for $P_{m,n}^\square$.*

Proof. Let $I_{-1}^\pi = \{(i'_t, j'_t)\}_{t=1}^{k-1}$ in such a way that $(i'_t, j'_t) \in \square(i_t, j_t, i_{t+1}, j_{t+1})$, for $t = 1, \dots, k-1$. We settle this result by induction on k. The case $k = 2$ follows from Theorem 3, so assume $k > 2$. Consider

$$\sum_{t=1}^{k-1} x_{i_t, j_t} - \sum_{t=1}^{k-2} x_{i'_t, j'_t} \quad \leq \quad 1, \tag{11}$$

$$x_{i_{k-1}, j_{k-1}} + x_{i_k, j_k} - x_{i'_{k-1}, j'_{k-1}} \quad \leq \quad 1. \tag{12}$$

Let $T := \square(1, 1, i_{k-1}, n)$ and $B := \square(i_{k-1}, 1, m, n)$. The inequality (11) induces a facet of $P_{i_{k-1},n}^\square$ by the inductive hypothesis. On the other hand, Theorem 3 implies that (12) induces a facet of $P_{m-i_{k-1}+1,n}^\square$. This implies that the hypotheses of Lemma 2 are satisfied, so $\pi x \leq \pi_0$ induces a facet of $P_{m,n}^\square$. $\square$

We may have $i_t = i_{t+1}$ or $j_t = j_{t+1}$ in Theorem 4 for any $t \in \{1, \dots, k-1\}$ (but not both, since this would contradict the fact that $\square(i_t, j_t, i_{t+1}, j_{t+1})$ contains exactly one pixel from I_{-1}^π). This implies that the family of facet-inducing inequalities specified by Theorem 4 includes the *interval constraints* $x_{i_1,j} - x_{i_2,j} + x_{i_3,j} - x_{i_4,j} + \cdots + x_{i_{2k+1},j} \leq 1$, with $j \in \{1, \dots, n\}$ and $i_t \in \{1, \dots, m\}$ for $t = 1, \dots, 2k+1$ such that $i_t < i_{t+1}$ for $t = 1, \dots, 2k$, coming from the full interval vectors polytope [9]. The results in [9] imply that these inequalities, together with the nonnegativity constraints, fully characterize $P_{m,n}^\square$ when $m = 1$.

Lemma 2 directly implies the following result.

Theorem 5. *Let $\pi \in \{-1,0,1\}^{mn}$. Let $p_1 = (i_1, j_1)$, $p_2 = (i_2, j_2)$, and $p_3 = (i_3, j_3)$ with $i_1 \leq i_2 \leq i_3$ and $j_1 < j_3 < j_2$, and assume $I_1^\pi = \{p_1, p_2, p_3\}$. Also assume $I_{-1}^\pi = \{q_1, q_2\}$ such that*

- *$I_{-1}^\pi \cap \square(p_1, p_2) \cap \square(p_1, p_3) = \{q_1\}$ and*
- *$I_{-1}^\pi \cap [\square(p_2, p_3) \backslash \square(p_1, p_2)] = \{q_2\}$,*

then $\pi x \leq 1$ is facet-inducing for $P_{m,n}^\square$ (see Figure 6).

Using the same proof technique as in Theorem 2, we can show the following generalization.

Theorem 6. *Let $\pi \in \{-1,0,1\}^{mn}$ with $|I_1^\pi| = |I_{-1}^\pi| + 1$. Assume that every pixel in I_0^π is reachable from some pixel in I_1^π. If there exists a list $L = \{R_1, \dots, R_k\}$ of rectangles such that*

(a) every pixel in I_{-1}^π is contained in R_i for some $i \in \{1, \dots, k\}$,

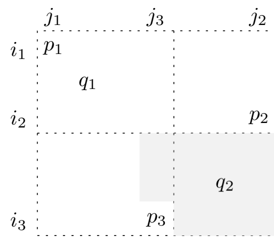


Figure 6: Structure considered in Theorem 5. The pixel q_2 can appear within the shaded area.

(b) $|R_i \cap I_1^\pi| = |R_i \cap I_{-1}^\pi| + 1$ *for* $i = 1, \ldots, k$, *and*

(c) $|I_{-1}^\pi \cap R_1| = 1$,

(d) $|I_{-1}^\pi \cap (R_i \setminus \bigcup_{j=1}^{i-1} R_j)| = 1$ *for* $i = 2, \ldots, k$,

then $\pi x \leq 1$ *is facet-inducing for* $P_{m,n}^{\square}$.

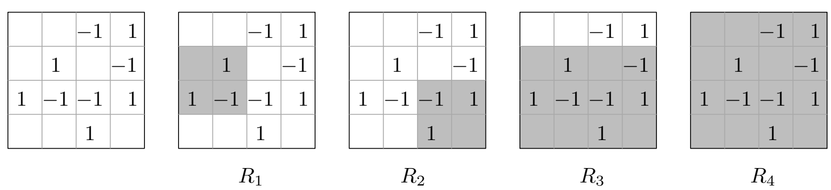


Figure 7: The coefficient vector π (in matrix form) of a valid inequality $\pi x \leq 1$ that verifies the hypotheses of Theorem 6. The gray rectangles in the subfigures represent the rectangles R_i of the list L of the hypothesis. Notice that the sets I_1^π and I_{-1}^π associated with this inequality do not satisfy the hypothesis (b) of Theorem 2.

When $k = 1$, the inequalities considered in Theorem 6 are exactly the inequalities studied in Theorem 2 with $|I_1^\pi| = 2$. For $k > 1$, each rectangle in the sequence $R_2, \ldots, R_k$ "adds" a variable with coefficient 1 and a variable with coefficient -1 to the inequality, both corresponding to pixels not included in the previous rectangles. Interestingly, hypotheses (a)-(d) of Theorem 6 seem to be necessary for facetness of inequalities $\pi x \leq 1$ with $\pi \in \{-1, 0, 1\}^{mn}$ having $|I_1^\pi| = |I_{-1}^\pi| + 1$ and such that every pixel in I_0^π is reachable from some pixel in I_1^π. We provide computational evidence of this fact in Section 5.

Theorems 3, 4, 5, and 6 provide facet-inducing inequalities satisfying $|I_1^\pi| = |I_{-1}^\pi| + 1$, a property shared with the interval constraints of the full interval vectors polytope. Theorem 7, on the other hand, provides facet-inducing inequalities with $|I_1^\pi| = |I_{-1}^\pi|$. The proof of this result relies on similar arguments to the ones given in the previous proofs, and is therefore omitted.

Theorem 7. *Let $\pi \in \{-1,0,1\}^{mn}$. Let $p_1 = (i_1, j_1)$, $p_2 = (i_2, j_2)$, and $p_3 = (i_3, j_3)$ with $i_1 \leq i_2 \leq i_3$ and $j_1 < j_3 < j_2$, and assume $I_1^\pi = \{p_1, p_2, p_3\}$. Also assume $I_{-1}^\pi = \{q_1, q_2, q_3\}$ such that*

- $I_{-1}^\pi \cap [\square(p_1, p_2) \backslash (\square(p_2, p_3) \cup \square(p_1, p_3))] = \{q_1\}$,
- $I_{-1}^\pi \cap [\square(p_2, p_3) \backslash (\square(p_1, p_2) \cup \square(p_1, p_3))] = \{q_2\}$,
- $I_{-1}^\pi \cap [\square(p_1, p_3) \backslash (\square(p_1, p_2) \cup \square(p_2, p_3))] = \{q_3\}$,

(see Figure 8) then $\pi x \leq 1$ is facet-inducing for $P_{m,n}^\square$.

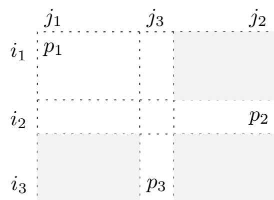


Figure 8: Structure considered in Theorem 7. The shaded areas correspond to the locations of the three negative coefficients.

This implies that $P_{m,n}^\square$ admits facets with $|I_1^\pi| = |I_{-1}^\pi|$. It would be interesting to explore whether this construction can be extended to $|I_1^\pi| > 3$.

4.2. Facet-inducing inequalities with coefficients in $\{-2,0,1\}$ and $\{-3,0,1\}$

We now explore valid inequalities with at least one coefficient greater than 1 in absolute value.

Theorem 8. *Let $\pi \in \{-2,0,1\}^{mn}$. If $I_1^\pi = \{(i_1, j_1), (i_2, j_2), (i_3, j_3)\}$ with $i_3 \leq i_1 < i_2$, $j_1 < j_2 \leq j_3$, and $I_{-2}^\pi = \{(i_1, j_2)\}$ (see Figure 9), then $\pi x \leq 1$ is facet-inducing for $P_{m,n}^\square$.*

Proof. Let $\bar{x} \in P_{m,n}^\square \cap \{0,1\}^{mn}$ be a feasible solution, and let $A = \{(i,j) \in P : \bar{x}_{ij} = 1\}$ be the set of pixels in $\bar{x}$. On the one hand, if $|A \cap I_1^\pi| \leq 1$, then $\pi\bar{x} \leq 1$ and the inequality is satisfied by $\bar{x}$. On the other hand, if any two pixels from I_1^π belong to A, then $I_{-2}^\pi \subseteq A$, so $\pi\bar{x} \leq 1$ and again the inequality is satisfied. Since $\bar{x}$ is an arbitrary feasible solution, then $\pi x \leq 1$ is a valid inequality.

Now for facetness. Assume $\lambda x = \lambda_0$ for every $x \in F^\pi$. We shall first show that $\lambda_r = 0$ for $r \in I_0^\pi$. If $i_3 < i_1$ and $j_2 < j_3$ then all pixels in I_0^π are reachable from some pixel in I_1^π, so $\lambda_r = 0$ for every $r \in I_0^\pi$ by Lemma 1.

So assume $i_3 = i_1$, hence $j_2 < j_3$. Let $C = \square(1, j_2, i_1 - 1, j_2)$ be the pixels located above (i_1, j_2). All pixels in $I_0^\pi \backslash C$ are reachable from some pixel in I_1^π, so $\lambda_r = 0$ for $r \in I_0^\pi \backslash C$ by Lemma 1. Now, for $t = 1, \ldots, i_1 - 1$, consider the solutions $\blacksquare(t, j_1, i_2, j_3)$ and $\blacksquare(t+1, j_1, i_2, j_3)$. Both satisfy $\pi x \leq 1$ with equality, and differ by the pixels $\square(t, j_1, t, j_3)$. Since $\lambda_r = 0$ for every

$r \in \square(t, j_1, t, j_3) \setminus \{(t, j_2)\}$, we conclude that $\lambda_{t,j_2} = 0$. This implies $\lambda_r = 0$ for every $r \in I_0^\pi$.

A similar argument settles the case $j_2 = j_3$. To conclude the proof, the solutions $\{\blacksquare((i_t, j_t))\}_{t=1}^3$ show $\lambda_{i_1 j_2} = \lambda_{i_2 j_2} = \lambda_{i_3 j_3}$ and, together with the solution $\blacksquare(i_3, j_1, i_2, j_3)$, they imply $\lambda_{i_1 j_2} = -2\lambda_{i_1 j_1}$. So $\pi x \leq 1$ induces a facet of $P_{m,n}^\square$. $\square$

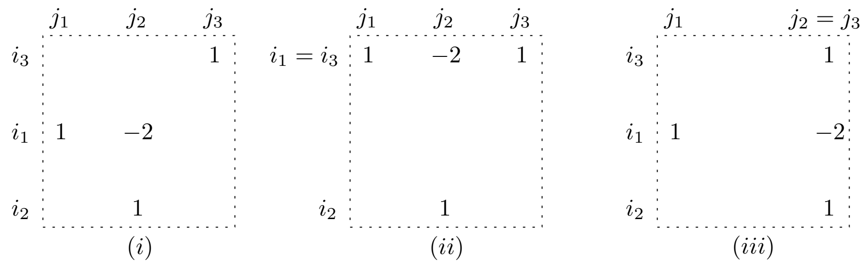


Figure 9: The three possible configurations for π in Theorem 8.

In our experiments with the PORTA [10] software package we observed that many facet-inducing inequalities have the structure present in the inequalities described in Theorem 8 and two additional variables with coefficients 1 and −1, respectively. The following lemma helps explain these inequalities and is an example of a *facet-preserving procedure*, namely a procedure that takes as input a facet-inducing inequality and "produces" a new inequality with larger support that is also facet-inducing if the hypotheses are satisfied.

Lemma 3. *Let $\pi \in \mathbb{Z}^{mn}$ such that $\pi x \leq 1$ is facet-inducing for $P_{m,n}^\square$. Let $p, q \in I_0^\pi$ and $\bar{\pi} x \leq 1$ be the inequality $\pi x + x_p - x_q \leq 1$. If*

(a) for every $S \subseteq I_{>0}^\pi$ such that $\pi \blacksquare(S) = 1$, we have $q \in \square(S \cup \{p\})$,

(b) every pixel in $I_0^{\bar{\pi}}$ is reachable from some pixel in $I_1^{\bar{\pi}}$,

(c) neither p nor q belong to $\square(I_{\neq 0}^\pi)$, and

(d) there exists $S \subseteq \square(I_{\neq 0}^\pi)$ with $\blacksquare(S) \in F^\pi$ such that $\blacksquare(S \cup \{p, q\}) \in F^\pi$,

then $\pi x + x_p - x_q \leq 1$ is also facet-inducing for $P_{m,n}^\square$.

Proof. We first show that $\pi x + x_p - x_q \leq 1$ is valid for $P_{m,n}^\square$. To this end, suppose there exists a feasible solution $\bar{x}$ with $\pi\bar{x} + \bar{x}_p - \bar{x}_q > 1$, and let $S \subseteq P$ be the rectangle represented by $\bar{x}$. Since $\pi\bar{x} \leq 1$ (due to the validity of $\pi x \leq 1$) and $\pi \in \mathbb{Z}^{mn}$, we have that $\pi\bar{x} = 1$ and $\bar{x}_p = 1$. This implies that $p \in S$, and by the hypothesis (a) we also have $q \in S$. Hence, $\bar{x}_q = 1$ and thus $\pi\bar{x} + \bar{x}_p - \bar{x}_q \leq 1$, which is a contradiction.

For facetness, assume $\lambda x = \lambda_0$ for every $x \in F^{\bar{\pi}}$. Lemma 1 together with the hypothesis (b) imply that $\lambda_r = 0$ for every $r \in I_0^{\bar{\pi}}$. Since $\pi x \leq 1$ induces a facet of $P^{\square}_{m,n}$, there exist $k := |I^{\pi}_{\neq 0}|$ affinely independent points $\bar{x}^1, \ldots, \bar{x}^k$ such that the system $\{\gamma \bar{x}^i = \gamma_0\}_{i=1}^k$ (together with $\gamma_r = 0$ for $r \in I^{\pi}_0$) only admits solutions of the form $\gamma = \alpha\pi$, for $\alpha \in \mathbb{R}$. For $i = 1, \ldots, k$, if $\bar{x}^i = \blacksquare(S^i)$, define $\tilde{x}^i = \blacksquare(S^i \cap I^{\pi}_{\neq 0})$, i.e., $\tilde{x}^i$ represents the solution obtained from $\bar{x}^i$ by restricting S^i to the smallest rectangle within S^i containing the pixels in $S^i \cap I^{\pi}_{\neq 0}$. None of these solutions includes the pixels p and q, by the hypothesis (c). This also implies that there exists $\alpha \in \mathbb{R}$ such that $\lambda_r = \alpha\pi_r = \alpha\bar{\pi}_r$ for every $r \in I^{\pi}_{\neq 0}$. Finally, the combination of $\tilde{x}^1$ and $\blacksquare(p)$ shows $\lambda_p = \alpha\bar{\pi}_p = \alpha$.

By the hypothesis (d), there exists $S \subseteq \square(I^{\pi}_{\neq 0})$ with $\blacksquare(S) \in F^{\pi}$ and $\blacksquare(S \cup \{p, q\}) \in F^{\pi}$, hence $\blacksquare(S \cup \{p, q\}) \in F^{\bar{\pi}}$. These two solutions, together with the observation that $\lambda_p = \alpha$, imply $\lambda_q = -\alpha$. This shows that $\lambda = \alpha\bar{\pi}$, so $\pi x + x_p - x_q \leq 1$ induces a facet of $P^{\square}_{m,n}$. □

Lemma 3 allows us to identify additional families of facet-inducing inequalities as, e.g., in the next result.

Corollary 2. *Let $\pi \in \{-2, 0, 1, -1\}^{mn}$. If $I^{\pi}_1 = \{(i_1, j_1), (i_2, j_2), (i_3, j_3), (i_4, j_4)\}$, $I^{\pi}_{-2} = \{(i_1, j_2)\}$, and $I^{\pi}_{-1} = \{(i_5, j_5)\}$ with $i_4 \leq i_5 \leq i_3 \leq i_1 < i_2$ and $j_4 \leq j_5 \leq j_1 < j_2 \leq j_3$, and such that $i_5 < i_3$ or $j_5 < j_1$ (see Figure 10 (i)), then $\pi x \leq 1$ is facet-inducing for $P^{\square}_{m,n}$.*

We now turn our attention to facet-inducing inequalities with at least one variable with coefficient -3.

Theorem 9. *Let $\pi \in \{-3, 0, 1\}^{mn}$. If $I^{\pi}_{-3} = \{(i, j)\}$ and $I^{\pi}_1 = \{(i, j_1), (i, j_2), (i_1, j), (i_2, j)\}$ with $i_1 < i < i_2$ and $j_1 < j < j_2$, then $\pi x \leq 1$ is facet-inducing for $P^{\square}_{m,n}$.*

Proof. The proof of validity goes along the same lines as the proof of Theorem 8, hence it is omitted. In order to verify facetness, assume $\lambda x = \lambda_0$ for every $x \in F^{\pi}$. Each pixel in I^{π}_0 is reachable from some pixel in I^{π}_1, hence $\lambda_r = 0$ for every $r \in I^{\pi}_0$ by Lemma 1. Call $q = (i, j)$, $p_1 = (i, j_1)$, $p_2 = (i, j_2)$, $p_3 = (i_1, j)$, and $p_4 = (i_2, j)$. The solutions $\{\blacksquare(p_i)\}_{i=1}^4$ show $\lambda_{p_1} = \lambda_{p_2} = \lambda_{p_3} = \lambda_{p_4}$ and, together with $\blacksquare(I^{\pi}_1)$, they imply $\lambda_q = -3\lambda_{p_1}$. This implies in turn that λ is a multiple of π, hence $\pi x \leq 1$ defines a facet of $P^{\square}_{m,n}$. □

Corollary 3. *Let $\pi \in \{-3, 0, 1, -1\}^{mn}$. If $I^{\pi}_{-3} = \{(i, j)\}$, $I^{\pi}_1 = \{(i, j_1), (i, j_2), (i_1, j), (i_2, j), (i_3, j_3)\}$, and $I^{\pi}_{-1} = \{(i_4, j_4)\}$ with $i_3 \leq i_4 \leq i_1 < i < i_2$ and $j_3 \leq j_4 \leq j_1 < j < j_2$, and such that $i_4 < i_1$ or $j_4 < j_1$, (see Figure 10 (ii)) then $\pi x \leq 1$ is facet-inducing for $P^{\square}_{m,n}$.*

It would be interesting to search for further configurations originating facet-inducing inequalities with such large coefficients. In our experiments with small instances, we could not find facet-inducing inequalities with (normalized) integer coefficients outside the range $\{-3, \ldots, 4\}$, and it could be relevant to explore whether this is the case in general.

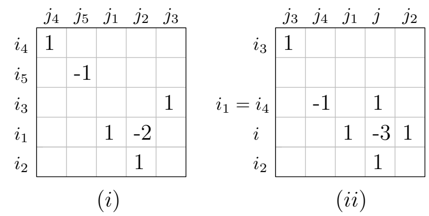


Figure 10: Valid inequalities of the form $\pi x \leq 1$ that verify the hypotheses of Corollary 2 (subfigure (*i*)) and Corollary 3 (subfigure (*ii*)).

5. Polyhedral computations

We report in this section the computational experiments we performed with the families of valid inequalities presented in the previous section.

The first set of experiments was designed in order to provide evidence of the fact that hypotheses (a)-(d) of Theorem 6 seem to be necessary for facetness of inequalities $\pi x \leq 1$ with $\pi \in \{-1, 0, 1\}^{mn}$ having $|I_1^\pi| = |I_{-1}^\pi| + 1$ and such that every pixel in I_0^π is reachable from some pixel in I_1^π. We performed an exhaustive computational verification of all facet-inducing inequalities of $P_{3,5}^\square$, $P_{4,4}^\square$ and $P_{3,6}^\square$, with the help of the PORTA software package. In all cases (approx. 7700 facet-inducing inequalities for $P_{3,5}^\square$, 9900 facet-inducing inequalities for $P_{4,4}^\square$, and 47500 facet-inducing inequalities for $P_{3,6}^\square$) a list L satisfying the hypotheses was indeed found. Due to impractical running times, larger instances could not be checked.

The main set of experiments, to be described in the remainder of this section, aimed at evaluating the practical effectiveness of the valid inequalities identified in Section 4. To this end, we generate inequalities from Theorems 2-9 and Corollaries 2-3, add them to the linear relaxation of the formulation (5)-(9), and study the improvement of the objective function achieved by this addition. These measurements provide computational evidence of the contribution of each family of inequalities to a cutting-plane based approach, and act as a proxy for their practical effectiveness in such a procedure.

5.1. Computational procedures for generating the inequalities

We first comment on the procedures generating the inequalities to be added to the linear relaxation, since the exhaustive generation of all inequalities was prohibitive in several cases.

Theorem 2: We enumerate all values $(i_1, i_2, i_3, j_1, j_2, j_3)$ with $1 \leq i_1 < i_2 < i_3 \leq m$ and $1 \leq j_1 < j_2 < j_3 \leq n$. For each such combination, let $p_1 = (i_1, j_1)$, $p_2 = (i_2, j_2)$, $p_3 = (i_3, j_3)$, and $R_1 = \square(p_1, p_2)$ and $R_2 = \square(p_1, p_3)$. Select at random two pixels $q_1 \in (R_1 \setminus R_2) \setminus \{p_2\}$ and $q_2 \in (R_2 \setminus R_1) \setminus \{p_3\}$ (see Figure 11). Let now $\pi \in \{-1, 0, 1\}^{mn}$ given by $I_1^\pi = \{p_1, p_2, p_3\}$ and $I_{-1}^\pi = \{q_1, q_2\}$. It is easy to see that the inequality $\pi x \leq 1$ satisfies the hypotheses of Theorem 2.

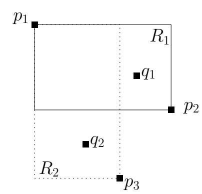


Figure 11: Structure of generated inequalities for Theorem 2.

Theorem 3: The inequalities described by this theorem are exhaustively added by considering all possible pairs of pixels $p_1 = (i_1, j_1)$ and $p_2 = (i_2, j_2)$ with $i_1 \leq i_2$ and $j_1 \leq j_2$, and all pixels $q \in \Box(p_1, p_2) \backslash \{p_1, p_2\}$. For each such set $\{p_1, p_2, q\}$, we construct the inequality $\pi x \leq 1$ with $\pi \in \{-1, 0, 1\}^{mn}$ given by $I_1^\pi = \{p_1, p_2\}$ and $I_{-1}^\pi = \{q\}$.
Theorem 4: For every $2 \leq k \leq m$, we generate the inequality $\pi x \leq 1$, $\pi \in \{-1, 0, 1\}^{mn}$, by considering $I_1^\pi = \{p_t\}_{t=1}^k$, with $p_t = (i_t, j_t)$, $i_t \leq i_{t+1}$, $j_t \leq j_{t+1}$ (all possible sequences are enumerated), and by defining $I_{-1}^\pi = \{q_1, q_2, \ldots, q_{k-1}\}$, where $q_i \in \Box(p_i, p_{i+1}) \backslash \{p_i, p_{i+1}\}$ (all possible values are also exhaustively enumerated). See Figure 12 for an illustration.

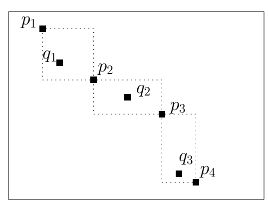


Figure 12: Structure of the generated inequalities for Theorem 4.

Theorem 5: For every set of three pixels $p_1 = (i_1, j_1)$, $p_2 = (i_2, j_2)$, and $p_3 = (i_3, j_3)$ with $i_1 \leq i_2 \leq i_3$ and $j_1 < j_3 < j_2$, we randomly pick pixels $q_1 = (i_4, j_4)$ and $q_2 = (i_5, j_5)$ in such a way that $i_1 \leq i_4 \leq i_2$, $j_1 \leq j_4 \leq j_3$, and $i_2 < i_5 \leq i_3$, $j_3 < j_5 \leq j_2$ (see Figure 6), thus ensuring that the hypotheses in Theorem 5 hold. In this setting, we generate the inequality $\pi x \leq 1$ with $\pi \in \{-1, 0, 1\}^{mn}$ given by $I_1^\pi = \{p_1, p_2, p_3\}$ and $I_{-1}^\pi = \{q_1, q_2\}$.
Theorem 6: We generate the list of rectangles $L = \{R_1, \ldots, R_k\}$ required by Theorem 6 as follows (see Figure 13). Rectangle R_1 is constructed by selecting an arbitrary pixel $q_1 = (i_1, j_1)$, and then pixels $p_1 = (i_2, j_1)$, $i_1 < i_2$ and $p_2 = (i_1, j_2)$, $j_1 < j_2$. We define $R_1 = \Box(p_1, p_2)$, and set $q_1 \in I_{-1}^\pi$ and $p_1, p_2 \in I_1^\pi$. Select now pixel $q_2 = (i_3, j_3)$ randomly, respecting the constraints $i_2 < i_3 \leq m$ and $j_2 < j_3 \leq n$. Pixels $p_3 = (i_4, j_3)$, $p_4 = (i_3, j_4)$ are selected as above, with $i_3 < i_4$ and $j_3 < j_4$. Define $R_2 = \Box(p_3, p_4)$, $q_2 \in I_{-1}^\pi$, and $p_3, p_4 \in I_1^\pi$. Finally, define rectangle $R_3 = \Box(i_1, j_1, i_4, j_4)$. We pick at random a pixel $q_3 = (i_5, j_5) \in \Box(i_1, j_3, i_3 - 1, j_4)$, and set $q_3 \in I_{-1}^\pi$. Notice that every rectangle

verifies the hypotheses of the theorem. At this point, we repeat the next steps: we select a vertex $q_4 = (i_6, j_6)$ for rectangle R_4 as we did with R_1, considering now $i_4 < i_6 \leq m$ and $j_5 < j_6 \leq n$, and performing the same sequence of steps as above. This continues until no more rectangles can be added, because we reach either the bottom-right pixel of the matrix or a predefined maximum value of k. The procedure is repeated for several starting pixels q_1, and for several values of k. Our goal with this procedure is to construct a set of inequalities with no intersection with the inequalities generated for Theorem 2 (which is generalized by Theorem 6).

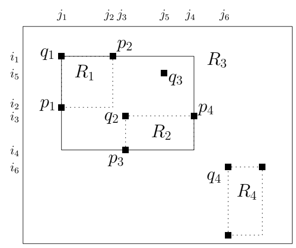


Figure 13: Structure of generated inequalities for Theorem 6.

Theorem 7: Analogously to the procedure for Theorem 2, each possible three-pixel set $P = \{p_1, p_2, p_3\}$, where $p_1 = (i_1, j_1)$, $p_2 = (i_2, j_2)$, and $p_3 = (i_3, j_3)$ with $i_1 \leq i_2 \leq i_3$ and $j_1 < j_3 < j_2$ is considered, and for each such set, three new pixels $Q = \{q_1, q_2, q_3\}$ satisfying the hypotheses for the negative coefficients are randomly selected. This yields the inequality $\pi x \leq 1$ with $\pi \in \{-1, 0, 1\}^{mn}$ given by $I_1^\pi = P$ and $I_{-1}^\pi = Q$.
Theorems 8 and 9: The inequalities for these theorems are exhaustively constructed, by enumerating all possible positions for placing pixel q with coefficient -2 (resp. -3), and then enumerating all possible candidate sets of pixels encircling q that satisfy the hypotheses.
Corollaries 2 and 3: The inequalities for Corollary 2 are constructed by considering each possible four-pixel set $P = \{p_1, p_2, p_3, q\}$, where $p_1 = (i_1, j_1)$, $p_2 = (i_2, j_2)$, $p_3 = (i_3, j_3)$, and $q = (i_1, j_2)$ with $i_3 \leq i_1 < i_2$ and $j_1 < j_2 \leq j_3$. We randomly select two more pixels $p_4 = (i_4, j_4)$ and $p_5 = (i_5, j_5)$, satisfying $i_4 \leq i_5 < i_3$ and $j_4 \leq j_5 < j_1$. In this setting, we generate the inequality $\pi x \leq 1$ with $\pi \in \{-2, -1, 0, 1\}^{mn}$ given by $I_1^\pi = \{p_1, p_2, p_3, p_4\}$, $I_{-2}^\pi = \{q\}$, and $I_{-1}^\pi = \{p_5\}$. The inequality $\pi x \leq 1$ satisfies the hypotheses of Corollary 2. Inequalities for Corollary 3 are added in a similar manner.

5.2. Computational results

We consider randomly-generated instances given by real-valued matrices with positive and negative coefficients, of different dimensions and densities (i.e., the proportion of nonzero entries). We employ Cplex 12.10 as the linear and integer programming solver. The experiments were carried out on a computer with an Intel Core i7-8550U CPU with 16 GB of RAM memory.

Table 1 reports our results on instances with known optima. For these cases, Cplex was able to produce an optimal solution for the integer problem within reasonable time limits. The columns "Instance - Size" and "Instance - Density" describe the instance characteristics. The column group "Int. model" contains the results of the complete solution of the integer programming formulation (4)-(9), reporting the objective function value in the column "Opt." and the total execution time in seconds in the column "T(s)". The column group "Linear relaxation" indicates that the solution value and execution time correspond to the linear relaxation of the formulation. The column group "Strengthened formulation" shows the results for the linear relaxation with the addition of the generated inequalities (we refer to this as the *strengthened formulation*). Within these last two groups, the column "Obj." reports the objective function value for the corresponding relaxation, the column "T(s)" reflects the execution time in seconds, and the column "Gap" reports the relative difference between the objective function value of the linear relaxation and the objective function value of the optimal integer solution. The column "Strengthened formulation - #Ineqs." contains the number of individual inequalities added in the strengthened formulation with respect to the initial model. Finally, the last column reports the reduction of the gap of the strengthened formulation with respect to the linear relaxation of the original formulation.

The results clearly show the effectiveness of the added inequalities: the optimal value is attained for 21% of the instances, every instance improves the solution value, and the average gap reduction is 91%. The number of generated inequalities quickly grows to large values, despite the fact that for several families only a fraction of all existing inequalities is constructed. Further, we remark that the generation of these inequalities does not depend on the input coefficients, only on the dimension of the input matrix. If these dimensions do not change, then the generated inequalities may be cached across several executions (i.e., we may store them in a pool, in order to not generate them again for a new instance with an input matrix with the same dimensions).

Tables 2 and 3 show the effectiveness of each indidivual family of valid inequalities. In Table 2, the column "Linear rel. Gap" reports the gap of the linear relaxation of the formulation, and the column group "Strengthened form." presents the gap of the linear relaxation of the strengthened formulation (column "Gap") and the number of inequalities added to the original model (column "# Ineqs."). The remaining columns report the gap and number of inequalities corresponding to the addition of each individual family of valid inequalities to the linear model. As this table shows, each family is effective in its own right. Although there are instances where the addition of some families does not lead to improvements, it is interesting to note that in several cases (e.g., instances 6×6, 0.6 and 10×10, 0.8) the combined inequalities yield a much smaller gap that any individual family of valid inequalities.

Table 3 summarizes the gap improvement of every family, compared to the linear relaxation. We give combined results of Corollaries 2 and 3 in the last two columns, since they are both consequences of Lemma 3. On average, the families given by Theorems 2, 5, 6, 7, 8, 9 and Corollaries 2 and 3 seem more effective, and facet families of Theorems 3, 4 and 6 less effective.

Instance		Int. model		Linear relaxation			Strengthened formulation				
Size	Density	Opt.	T(s)	Obj.	T(s)	Gap	Obj.	T(s)	Gap	#Ineqs.	Gap red.
5×5	0.2	297	0.024	343	0.001	15.5%	297	0.015	0.0%	3306	15.5%
	0.4	314	0.016	358	0.001	14.0%	279	0.017	0.0%		14.0%
	0.6	170	0.048	244	0.001	43.5%	180	0.011	5.9%		37.7%
	0.8	292	0.036	440	0.001	50.7%	344	0.008	17.8%		32.9%
6×6	0.2	313	0.033	383	0.001	22.4%	313	0.028	0.0%	9980	22.4%
	0.4	179	0.016	296	0.001	65.4%	207	0.024	15.6%		49.7%
	0.6	158	0.014	237	0.001	50.0%	191	0.027	20.9%		29.1%
	0.8	705	0.014	894	0.002	26.8%	592	0.038	0.0%		26.8%
7×7	0.2	438	0.116	660	0.002	50.7%	438	0.096	0.0%	25310	50.7%
	0.4	289	0.137	552	0.003	91.0%	347	0.092	20.1%		70.9%
	0.6	856	0.208	960	0.005	12.2%	746	0.111	0.0%		12.2%
	0.8	510	0.116	807	0.002	58.2%	547	0.099	7.3%		51.0%
8×8	0.2	264	0.118	659	0.004	149.6%	405	0.208	53.4%	56544	96.2%
	0.4	569	0.127	923	0.004	62.2%	616	0.238	8.3%		54.0%
	0.6	398	0.13	669	0.004	68.1%	442	0.166	11.1%		57.0%
	0.8	294	0.119	751	0.004	155.4%	467	0.238	58.8%		96.6%
9×9	0.2	756	0.312	1118	0.013	47.9%	743	0.657	0.0%	114851	47.9%
	0.4	405	0.302	867	0.014	114.1%	584	0.626	44.2%		69.9%
	0.6	674	0.341	1295	0.014	92.1%	756	0.644	12.2%		80.0%
	0.8	782	0.3	1314	0.013	68.0%	796	0.625	1.8%		66.2%
10×10	0.2	1403	0.451	1639	0.037	16.8%	1202	1.804	0.0%	216385	16.8%
	0.4	333	0.315	1027	0.013	208.4%	600	1.739	80.2%		128.2%
	0.6	282	0.451	1131	0.011	301.1%	639	1.599	126.6%		174.5%
	0.8	584	0.449	1108	0.019	89.7%	648	1.759	11.0%		78.8%
11×11	0.2	1131	0.945	1869	0.049	65.3%	1179	4.433	4.2%	383776	61.0%
	0.4	960	0.925	1627	0.035	69.5%	962	4.035	0.2%		69.3%
	0.6	1314	0.929	1807	0.046	37.5%	1209	4.798	0.0%		37.5%
	0.8	503	0.813	1335	0.033	165.4%	764	3.724	51.9%		113.5%
12×12	0.2	1101	2.529	1905	0.057	73.0%	1171	9.516	6.4%	647462	66.7%
	0.4	1129	2.856	1939	0.055	71.7%	1215	10.518	7.6%		64.1%
	0.6	806	1.873	1681	0.047	108.6%	980	7.914	21.6%		87.0%
	0.8	356	3.435	1701	0.049	377.8%	890	8.053	150.0%		227.8%
13×13	0.2	672	7.694	2166	0.045	222.3%	1145	15.82	70.4%	1047556	151.9%
	0.4	969	3.736	1944	0.047	100.6%	1091	14.79	12.6%		88.0%
	0.6	692	3.576	1756	0.046	153.8%	1050	15.291	51.7%		102.0%
	0.8	443	2.961	1562	0.059	252.6%	860	14.288	94.1%		158.5%
14×14	0.2	922	14.644	2644	0.112	186.8%	1427	30.23	54.8%	1635438	132.0%
	0.4	961	13.648	2427	0.09	152.6%	1336	24.046	39.0%		113.5%
	0.6	992	7.925	2261	0.076	127.9%	1218	27.211	22.8%		105.1%
	0.8	778	11.156	2389	0.102	207.1%	1284	27.404	65.0%		142.0%
15×15	0.2	2158	16.718	3256	0.167	50.9%	2034	68.618	0.0%	2476006	50.9%
	0.4	616	27.127	2457	0.131	298.9%	1259	46.356	104.4%		194.5%
	0.6	1005	30.915	3067	0.11	205.2%	1620	47.3	61.2%		144.0%
	0.8	954	27.071	2791	0.11	192.6%	1530	48.59	60.4%		132.2%
16×16	0.2	455	25.754	2582	0.106	467.5%	1334	61.779	193.2%	3649520	274.3%
	0.4	1078	104.351	3430	0.132	218.2%	1831	76.636	69.9%		148.3%
	0.6	856	75.323	3105	0.149	262.7%	1599	70.726	86.8%		175.9%
	0.8	963	25.732	3082	0.158	220.0%	1566	61.733	62.6%		157.4%

Table 1: Results for random instances with known optima.

Instance		Linear rel.	Strengthened form.		Theorem 2		Theorem 3		Theorem 4		Theorem 5		Theorem 6		Theorem 7		Theorem 8		Theorem 9		Cors. 2 and 3	
Size	Dens.	Gap	Gap	# Ineqs.	Gap	#Ineqs.	Gap	#Ineqs.	Gap	#Ineqs.	Gap	#Ineqs.	Gap	#Ineqs.	Gap	#Ineqs.	Gap	#Ineqs.	Gap	#Ineqs.	Gap	#Ineqs.
5 × 5	0.2	15.5%	0.0%	3306	10.8%	40	6.1%	800	0.0%	1619	0.0%	200	12.1%	164	11.8%	100	3.7%	400	6.4%	100	15.5%	14
	0.4	14.0%	0.0%		13.7%		0.0%		0.0%		0.0%		10.2%		7.3%		9.2%		13.4%		14.0%	
	0.6	43.5%	5.9%		43.5%		26.5%		22.9%		40.0%		41.2%		42.4%		23.5%		23.5%		43.5%	
	0.8	50.7%	17.8%		50.7%		40.8%		24.3%		50.7%		25.7%		50.7%		38.7%		46.2%		50.7%	
6 × 6	0.2	22.4%	0.0%	9980	9.3%	200	9.6%	2290	9.6%	4673	1.9%	700	16.3%	629	3.8%	400	4.2%	1225	11.8%	400	20.8%	21
	0.4	65.4%	15.6%		65.4%		64.3%		64.3%		59.8%		59.2%		62.0%		35.2%		34.6%		65.4%	
	0.6	50.0%	20.9%		48.7%		50.0%		50.0%		38.0%		50.0%		35.4%		33.5%		36.1%		50.0%	
	0.8	26.8%	0.0%		25.0%		16.0%		0.0%		4.7%		22.1%		24.4%		11.2%		18.7%		24.3%	
7 × 7	0.2	50.7%	0.0%	25310	46.6%	700	28.3%	5537	28.3%	11336	17.8%	1960	33.8%	2057	43.4%	1225	19.2%	3136	28.5%	1225	43.2%	153
	0.4	91.0%	20.1%		87.9%		85.5%		85.5%		55.0%		81.0%		81.7%		44.6%		41.2%		85.8%	
	0.6	12.2%	0.0%		11.8%		5.1%		0.0%		7.6%		5.0%		12.0%		10.8%		10.8%		10.8%	
	0.8	58.2%	7.3%		46.7%		32.4%		32.4%		11.4%		47.1%		50.8%		40.8%		35.3%		41.8%	
8 × 8	0.2	149.6%	53.4%	56544	126.9%	1960	149.6%	11872	149.6%	24366	94.3%	4704	147.7%	5700	122.0%	3136	76.5%	7056	68.6%	3136	125.8%	688
	0.4	62.2%	8.3%		56.2%		50.6%		48.7%		30.1%		52.0%		55.0%		23.6%		33.7%		56.8%	
	0.6	68.1%	11.1%		57.0%		68.1%		62.3%		24.9%		68.1%		42.5%		39.5%		40.2%		48.0%	
	0.8	155.4%	58.8%		130.6%		132.7%		132.7%		94.9%		145.6%		124.8%		85.4%		92.2%		108.5%	
9 × 9	0.2	47.9%	0.0%	114851	32.9%	4704	37.7%	23256	34.4%	47766	6.4%	10080	44.3%	14000	22.2%	7056	20.1%	14400	18.7%	7056	31.2%	2286
	0.4	114.1%	44.2%		95.6%		110.9%		110.9%		66.4%		105.7%		92.4%		81.2%		79.0%		93.3%	
	0.6	92.1%	12.2%		66.2%		76.4%		76.4%		36.7%		71.5%		70.8%		42.7%		40.7%		53.9%	
	0.8	68.0%	1.8%		57.3%		46.0%		46.0%		28.6%		46.0%		55.2%		16.6%		27.1%		47.3%	
10 × 10	0.2	16.8%	0.0%	216385	16.3%	10080	6.6%	42450	1.6%	87238	8.5%	19800	10.5%	30265	12.6%	14400	9.8%	27225	14.3%	14400	13.8%	6180
	0.4	208.4%	80.2%		154.7%		207.2%		206.9%		112.3%		207.2%		136.9%		142.3%		119.2%		151.4%	
	0.6	301.1%	126.6%		214.9%		301.1%		301.1%		176.2%		283.7%		200.7%		167.4%		151.4%		222.7%	
	0.8	89.7%	11.0%		57.2%		80.5%		80.3%		35.5%		87.5%		54.3%		52.2%		43.2%		53.6%	
11 × 11	0.2	65.3%	4.2%	383776	47.5%	19800	54.8%	73205	54.8%	150420	16.4%	36300	56.8%	60765	47.4%	27225	33.4%	48400	31.5%	27225	38.2%	14546
	0.4	69.5%	0.2%		47.8%		55.2%		54.6%		15.8%		60.2%		42.1%		35.7%		36.3%		38.0%	
	0.6	37.5%	0.0%		26.6%		27.1%		27.1%		5.7%		29.0%		25.7%		14.0%		15.7%		22.9%	
	0.8	165.4%	51.9%		118.7%		162.4%		162.4%		79.3%		155.9%		103.6%		94.6%		84.3%		101.8%	
12 × 12	0.2	73.0%	6.4%	647462	46.3%	36300	71.6%	120472	71.6%	247504	23.0%	62920	68.5%	113296	41.7%	48400	33.7%	81796	32.3%	48400	40.7%	30746
	0.4	71.7%	7.6%		57.0%		71.7%		70.0%		20.6%		71.7%		51.3%		34.3%		33.7%		40.1%	
	0.6	108.6%	21.6%		73.6%		108.6%		108.6%		49.8%		105.6%		70.7%		65.1%		38.1%		58.8%	
	0.8	377.8%	150.0%		248.9%		377.8%		367.7%		221.6%		377.8%		235.7%		247.2%		194.4%		218.8%	
13 × 13	0.2	222.3%	70.4%	1047556	151.9%	62920	222.3%	190632	213.4%	391457	116.5%	104104	222.3%	199844	141.2%	81796	124.7%	132496	103.4%	81796	130.5%	59798
	0.4	100.6%	12.6%		58.4%		100.6%		100.2%		34.9%		100.6%		53.5%		42.1%		34.8%		41.4%	
	0.6	153.8%	51.7%		105.9%		153.8%		153.3%		73.6%		150.4%		101.5%		91.9%		87.6%		91.2%	
	0.8	252.6%	94.1%		157.6%		252.6%		250.3%		141.8%		252.6%		153.1%		147.0%		127.8%		150.6%	
14 × 14	0.2	186.8%	54.8%	1635438	136.1%	104104	184.1%	291746	183.4%	598827	95.2%	165620	185.4%	335545	133.7%	132496	98.9%	207025	84.3%	132496	96.5%	108868
	0.4	152.6%	39.0%		91.8%		152.6%		152.1%		70.2%		152.6%		91.7%		89.1%		65.7%		78.3%	
	0.6	127.9%	22.8%		64.5%		127.9%		127.8%		55.2%		127.9%		62.1%		65.2%		47.9%		47.7%	
	0.8	207.1%	65.0%		140.1%		207.1%		207.1%		104.8%		207.1%		132.0%		127.0%		97.6%		114.0%	
15 × 15	0.2	50.9%	0.0%	2476006	39.0%	165620	50.9%	433825	50.9%	889924	5.4%	254800	50.1%	541013	36.1%	207025	15.1%	313600	19.1%	207025	20.3%	187702
	0.4	298.9%	104.4%		175.8%		298.9%		288.6%		167.9%		296.4%		177.4%		173.5%		130.8%		165.8%	
	0.6	205.2%	61.2%		128.6%		185.3%		185.2%		91.3%		183.4%		128.6%		118.2%		84.3%		103.3%	
	0.8	192.6%	60.4%		124.6%		191.7%		191.7%		94.8%		189.8%		123.5%		122.4%		99.6%		102.4%	
16 × 16	0.2	467.5%	193.2%	3649520	293.2%	254800	467.5%	629120	467.5%	1289838	280.0%	380800	467.5%	842540	295.0%	313600	282.2%	462400	216.7%	313600	259.8%	309401
	0.4	218.2%	69.9%		151.8%		218.2%		217.4%		116.6%		215.3%		145.9%		124.8%		98.3%		106.4%	
	0.6	262.7%	86.8%		173.8%		262.7%		262.4%		146.6%		261.2%		165.0%		163.1%		118.2%		129.0%	
	0.8	220.0%	62.6%		125.2%		215.1%		215.1%		110.1%		214.8%		113.5%		115.6%		92.9%		99.8%	

Table 2: Effectiveness of individual families of valid inequalities.

Instance		Gap improvement of each family								
Size	Density	Thm. 2	Thm. 3	Thm. 4	Thm. 5	Thm. 6	Thm. 7	Thm. 8	Thm. 9	Cors. 2 and 3
5 × 5	0.2	4.7%	9.4%	15.5%	15.5%	3.4%	3.7%	11.8%	9.1%	0.0%
	0.4	0.3%	14.0%	14.0%	14.0%	3.8%	6.7%	4.8%	0.6%	0.0%
	0.6	0.0%	17.1%	20.6%	3.5%	2.4%	1.2%	20.0%	20.0%	0.0%
	0.8	0.0%	9.9%	26.4%	0.0%	25.0%	0.0%	12.0%	4.5%	0.0%
6 × 6	0.2	13.1%	12.8%	12.8%	20.5%	6.1%	18.5%	18.2%	10.5%	1.6%
	0.4	0.0%	1.1%	1.1%	5.6%	6.2%	3.4%	30.2%	30.7%	0.0%
	0.6	1.3%	0.0%	0.0%	12.0%	0.0%	14.6%	16.5%	13.9%	0.0%
	0.8	1.8%	10.8%	26.8%	22.1%	4.7%	2.4%	15.6%	8.1%	2.6%
7 × 7	0.2	4.1%	22.4%	22.4%	32.9%	16.9%	7.3%	31.5%	22.2%	7.5%
	0.4	3.1%	5.5%	5.5%	36.0%	10.0%	9.3%	46.4%	49.8%	5.2%
	0.6	0.4%	7.0%	12.2%	4.6%	7.1%	0.1%	1.4%	1.4%	1.4%
	0.8	11.6%	25.9%	25.9%	46.9%	11.2%	7.5%	17.5%	22.9%	16.5%
8 × 8	0.2	22.7%	0.0%	0.0%	55.3%	1.9%	27.7%	73.1%	81.1%	23.9%
	0.4	6.0%	11.6%	13.5%	32.2%	10.2%	7.2%	38.7%	28.5%	5.5%
	0.6	11.1%	0.0%	5.8%	43.2%	0.0%	25.6%	28.6%	27.9%	20.1%
	0.8	24.8%	22.8%	22.8%	60.5%	9.9%	30.6%	70.1%	63.3%	46.9%
9 × 9	0.2	15.0%	10.2%	13.5%	41.5%	3.6%	25.7%	27.8%	29.2%	16.7%
	0.4	18.5%	3.2%	3.2%	47.7%	8.4%	21.7%	32.8%	35.1%	20.7%
	0.6	26.0%	15.7%	15.7%	55.5%	20.6%	21.4%	49.4%	51.5%	38.3%
	0.8	10.7%	22.0%	22.0%	39.4%	22.0%	12.8%	51.4%	40.9%	20.7%
10 × 10	0.2	0.5%	10.3%	15.2%	8.3%	6.3%	4.2%	7.0%	2.5%	3.0%
	0.4	53.8%	1.2%	1.5%	96.1%	1.2%	71.5%	66.1%	89.2%	57.1%
	0.6	86.2%	0.0%	0.0%	124.8%	17.4%	100.4%	133.7%	149.7%	78.4%
	0.8	32.5%	9.3%	9.4%	54.3%	2.2%	35.5%	37.5%	46.6%	36.1%
11 × 11	0.2	17.8%	10.4%	10.4%	48.9%	8.5%	17.9%	31.8%	33.8%	27.1%
	0.4	21.7%	14.3%	14.9%	53.7%	9.3%	27.4%	33.8%	33.2%	31.5%
	0.6	11.0%	10.4%	10.4%	31.8%	8.5%	11.8%	23.5%	21.8%	14.6%
	0.8	46.7%	3.0%	3.0%	86.1%	9.5%	61.8%	70.8%	81.1%	63.6%
12 × 12	0.2	26.7%	1.5%	1.5%	50.1%	4.5%	31.3%	39.3%	40.7%	32.3%
	0.4	14.8%	0.0%	1.8%	51.1%	0.0%	20.5%	37.5%	38.1%	31.6%
	0.6	35.0%	0.0%	0.0%	58.8%	3.0%	37.8%	43.4%	70.5%	49.8%
	0.8	128.9%	0.0%	10.1%	156.2%	0.0%	142.1%	130.6%	183.4%	159.0%
13 × 13	0.2	70.4%	0.0%	8.9%	105.8%	0.0%	81.1%	97.6%	118.9%	91.8%
	0.4	42.2%	0.0%	0.4%	65.7%	0.0%	47.2%	58.5%	65.8%	59.2%
	0.6	47.8%	0.0%	0.4%	80.2%	3.3%	52.3%	61.9%	66.2%	62.6%
	0.8	95.0%	0.0%	2.3%	110.8%	0.0%	99.6%	105.6%	124.8%	102.0%
14 × 14	0.2	50.7%	2.7%	3.4%	91.5%	1.4%	53.0%	87.9%	102.5%	90.2%
	0.4	60.8%	0.0%	0.4%	82.3%	0.0%	60.9%	63.5%	86.9%	74.3%
	0.6	63.4%	0.0%	0.1%	72.7%	0.0%	65.8%	62.7%	80.0%	80.2%
	0.8	67.0%	0.0%	0.0%	102.3%	0.0%	75.1%	80.1%	109.5%	93.1%
15 × 15	0.2	11.9%	0.0%	0.0%	45.5%	0.8%	14.7%	35.8%	31.8%	30.6%
	0.4	123.1%	0.0%	10.2%	131.0%	2.4%	121.4%	125.3%	168.0%	133.1%
	0.6	76.6%	19.9%	20.0%	113.8%	21.8%	76.6%	87.0%	120.9%	101.9%
	0.8	67.9%	0.8%	0.8%	97.8%	2.7%	69.1%	70.1%	93.0%	90.2%
16 × 16	0.2	174.3%	0.0%	0.0%	187.5%	0.0%	172.5%	185.3%	250.8%	207.7%
	0.4	66.4%	0.0%	0.8%	101.6%	2.9%	72.3%	93.4%	119.9%	111.8%
	0.6	88.9%	0.0%	0.4%	116.1%	1.5%	97.8%	99.7%	144.5%	133.8%
	0.8	94.8%	5.0%	5.0%	110.0%	5.3%	106.5%	104.5%	127.1%	120.3%
Average:		38.6%	6.5%	8.6%	63.0%	6.0%	43.2%	55.7%	65.7%	49.9%

Table 3: Gap improvement of individual families of valid inequalities.

6. Conclusions

We present in this work a polyhedral study of the polytope associated with a natural integer programming formulation for the maximum subarray problem for $d = 2$. Our objective is to identify strong families of valid inequalities that could be useful within a cutting plane procedure, or within a linear programming-based rounding heuristic for this problem. The final goal of this analysis is to obtain a strong column generation algorithm for RPC.

This article introduces several families of facet-inducing inequalities, many of them with coefficients in $\{-1, 0, 1\}$. From a polyhedral point of view, it would be desirable to achieve a more thorough theoretical treatment of these inequalities as, e.g., providing necessary and sufficient conditions ensuring facetness for general valid inequalities with coefficients in $\{-1, 0, 1\}$. We believe that Theorem 6 yields a promising basis for such a characterization, and proof of the remaining necessary condition could be addressed in a future work.

In addition, our computational results provide evidence of the effectiveness of the presented inequalities in the reduction of the dual bound for the proposed formulation.

From a practical point of view, the computational complexity of the separation problems associated with the introduced families is of interest, in particular since exhaustive enumerations do not provide polynomial-time algorithms for all of these problems and, furthermore, may not be practical in medium- to large-sized instances. The design of fast heuristics for separating these families could be of practical interest as well.

Acknowledgment. We are indebted to the anonymous reviewers for their thorough comments and suggestions.

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