Optimal strategies in a production-inventory control model

Pablo Azcue*, Esther Frostig† and Nora Muler*

Abstract

We consider a production-inventory control model with finite capacity and two different production rates, assuming that the cumulative process of customer demand is given by a compound Poisson process. It is possible at any time to switch over from the different production rates but it is mandatory to switch-off when the inventory process reaches the storage maximum capacity. We consider holding, production, shortage penalty and switching costs. This model was introduced by Doshi, Van Der Duyn Schouten and Talman in 1978. Our aim is to minimize the expected discounted cumulative costs up to infinity over all admissible switching strategies. We show that the optimal cost functions for the different production rates satisfy the corresponding Hamilton-Jacobi-Bellman system of equations in a viscosity sense and prove a verification theorem. The way in which the optimal cost functions solve the different variational inequalities gives the switching regions of the optimal strategy, hence it is stationary in the sense that depends only on the current production rate and inventory level. We define the notion of finite band strategies and derive, using scale functions, the formulas for the different costs of the band strategies with one or two bands. We also show that there are examples where the switching strategy presented by Doshi et al. is not the optimal strategy.

Key words. production-inventory model, optimal switching strategies, compound Poisson process, scale functions, HJB equation, viscosity solutions.

1. Introduction

The classical production-inventory model considers a single machine that produces a certain product. Finished products are stored and the storage capacity can be finite or infinite. Moreover, the classical model assume a constant production rate, customer demands arriving according to a Poisson process and size demands distributed as i.i.d random variables. When the stock on hand is less than the demand then either the excess of the demand is lost or backlogged. In the first case the inventory level is always positive, while in the latter it can be negative. The costs associated with this model are holding cost and lost-sales cost. Higher production rates yield fewer lost-sale cost but higher holding cost and viceversa. Thus, there is a trade-off between holding and lost-sales costs. Therefore, researchers have looked for the optimal strategy to minimize the expected cost. One of the prominent strategy discussed in the literature is the two regime switching policy. Under this policy, the production rate switches from high to low rate when the inventory increases above a given level y_1; also, the production rate switches from low to high rate when the inventory becomes smaller than a given level y_2, where $y_2 < y_1$.

In the operations research literature, most articles have considered the average cost per time unit assuming that the system is at steady state. Gavish and Graves [13] and Gavish and Keilson [14] studied the case where once the inventory level reaches a given threshold y_1, productions stops; and production resumes when the inventory level down-crosses another threshold y_2, where $y_2 < y_1$. In these two papers, customers arrive according to Poisson process and backlogging is permitted. In the first paper the demand is always for one item and the machine produces one item per time unit, and in the second one the demand is exponentially distributed. In both papers, the average cost per time unit is obtained. De Kok, Tijms and Van Der Schouten [8], De Kok [9] and De Kok [10] studied an infinite capacity production inventory system where demand occurs according to a compound Poisson process and unsatisfied demand is backlogged. They considered two production rates $\sigma_2 < \sigma_1$ where the production rate is switched to σ_2 once the inventory level is above y_1 and it is switched back to σ_1 when the inventory level down-crosses y_2. In the first paper, unsatisfied demand is backlogged and in the second one, unsatisfied demand is lost. Performance measures that are considered under some constrains on the switching and holding costs are: the average amount of stock-out per unit time, the fraction of

*Departamento de Matematicas, Universidad Torcuato Di Tella. Argentina.
†Department of Statistics, University of Haifa. Israel.

demand to be met directly from stock on hand (in the backlog case) and the average amount of lost sales. Doshi, Van Der Duyn and Talman [11] considered a finite capacity production inventory model with lost sales and similar production rate policy as in [8], [9] and [10]. They obtained the steady-state distribution of the inventory level for this model and hence the average cost per time unit.

More recently, Shi, Katehakis, Melamed and Xia [21] considered an infinite capacity production-inventory model with compound Poisson demand, lost sales and constant production rate. They obtained the expected discounted cost and then the production rate that minimizes it. Barron, Perry and Stadje [4] considered the model of Doshi et al. [11] under the assumption of Markov additive arrival process and phase-type demand and obtained the expected discounted cost.

The optimal two-regime switching problem, also called starting-and-stopping problem, has been studied extensively, in the diffusion setting and some special profit functions, Brekke and Oksendal [6] apply a verification approach for solving the variational inequality associated with this impulse control problem. Pham and Vath [18], Hamadène and Jeanblanc [15], and Bayraktar and Egami [5] between others, studied various extensions of this model. Also in the diffusion setting, Pham, Vath and Zhou [19] considered the case of multiple-regime switching. Azcue and Muler [3] studied a mixed singular control/switching problem for multiple regimes in the compound Poisson setting.

The rest of the paper is structured as follows. Section 2 describes the model setup and some basic results are derived in Section 3. In Section 4, we show that the optimal cost functions for the different production rates satisfy the corresponding Hamilton-Jacobi-Bellman system of equations in a viscosity sense and prove both characterization and verification results. Moreover, we prove that there exists an optimal production-inventory strategy and that it has a band structure. In Section 5, we introduce the concept of finite band strategies depending on the number of connected components of the non-action regions; and in Section 6 we use the scale functions to find the formulas of the holding, shortage and switching cost functions for the band strategies with one or two connected components. Finally, in Section 7, we identify the optimal strategies and the corresponding cost functions for a number of concrete examples with exponentially distributed customer demands.

2. Model

In this paper we address a production-inventory control model with finite storage capacity $b > 0$ and two production rates: σ_1 and σ_2 such that $0 < \sigma_2 < \sigma_1$; this model was introduced by Doshi et al. [11]. We say that the production is in phase $i = 1, 2$ when the production rate is σ_i, whenever the inventory level reaches level b, the production is stopped i.e. $\sigma_0 = 0$ at inventory level b. We assume that the cumulative process of customer demand is given by the compound Poisson process

$$\sum_{n=1}^{N_t} Y_n,$$

where N_t is a Poisson process with rate of arrival λ and the size of the demand Y_n are i.i.d positive random variables with distribution F and finite mean. Let us call τ_n the arrival of customer demand n. We also assume that $l \leq 0$ is the minimum level below which the inventory is not allowed to decrease. If the inventory drops below l the part of the demand below l is lost and production resumes at inventory level l.

The following costs are considered:

- **Holding and production costs**. $h_i : [l, b) \to [0, \infty)$ for $i = 1, 2$ correspond to the holding and production cost in phase i when the inventory level is $x \in [l, b)$. We assume that it is bounded with finitely many discontinuities and Lipschitz between discontinuities with Lipschitz constant m_h. $h_0(b) \geq 0$ corresponds to the holding cost at inventory level b.
- **Shortage penalty costs**. $p : [0, \infty) \to [0, \infty)$ corresponds to the penalty function cost when the amount y of the demand of a customer is lost. We assume that it is non-negative and non-decreasing. Moreover,

$$\int_0^\infty p(y) dF(y) < \infty. \tag{2.1}$$

- **Switching costs**. K_{ij} corresponds to the fixing cost of switching from phase i to phase j where i, $j = 0, 1, 2$. Here we include the costs of switch on (K_{0i} where $i = 1, 2$) and the costs of switch off (K_{i0} where $i = 1, 2$) the production process when the inventory reaches level b. We add the following conditions on the switching costs in order to penalize simultaneous changes of phases:

$$K_{0i} \le K_{0j} + K_{ji} \text{ for } \{i,j\} = \{1,2\}, \quad K_{12} + K_{21} > 0. \tag{2.2}$$

Remark 2.1. *We assume here that it is possible at any time to switch over from phase i to phase j where $1 \le i, j \le 2$ but it is mandatory to switch off (namely to go to phase 0) when the inventory process reaches level b. On top of that, the phase should be 1 or 2 (that corresponds to positive production rate) whenever the inventory process is in the interval $[l, b)$. Moreover, if a demand arrives and the inventory level before this arrival minus the demand of the customer is less than the backlog $l \le 0$, this demand is covered up to l paying the corresponding penalty cost of the part of the demand that has been lost given by function p .*

Our aim is to minimize the expected discounted cumulative costs over all possible production strategies. A production strategy can be defined as $\pi = (T_k, J_k)_{k\ge1}$ where T_k are the switching times from phase J_{k-1} to phase J_k and $J_k \in \{0, 1, 2\}$. We call $T_0 = 0$ and J_0 as the initial phase. In addition, we assume that $T_1 < T_2 < T_3 < \cdots$, and $J_k \ne J_{k-1}$.

Given a initial inventory level x, an initial phase $J_0 = i$ and a production strategy $\pi = (T_k, J_k)_{k\ge1}$, the controlled process is defined recursively as $X^\pi_{T_0} = x$, $T_0 = 0$, and

$$X^\pi_t = X^\pi_{T_k} + \sigma_{J_k}(t - T_k) - \sum\nolimits_{n=N_{T_k}}^{N_t} \min\{Y_n, X^\pi_{\tau_n^-} - l\} \text{ for } t \in [T_k, T_{k+1}). \tag{2.3}$$

Let us define the auxiliary inventory process,

$$\overset{\vee}{X}{}^\pi_t := X^\pi_t \text{ for } t \ne \tau_n \text{ and } \overset{\vee}{X}{}^\pi_{\tau_n} = X^\pi_{\tau_n^-} - Y_n,$$

so $X^\pi_{\tau_n} = l \vee \overset{\vee}{X}{}^\pi_{\tau_n}$, this corresponds to the controlled process before it eventually resumes at inventory level l.

Let us also define the controlled phase process

$$\mathcal{J}_t := J_k \text{ for } t \in [T_k, T_{k+1}). \tag{2.4}$$

A production strategy $\pi = (T_k, J_k)_{k\ge0} \in \Pi_{x,i}$ starting at phase i and inventory level x is *admissible* if it is $\mathcal{F}_t$-adapted, càdlàg and satisfies,

- $T_0 = 0$ and $J_0 = i$.
- If the current inventory level is less than b, then the phase should be either 1 or 2. More precisely, if $X^\pi_t < b$ then $\mathcal{J}_{t^-}$ must be 1 or 2.
- If at time t, the phase process $\mathcal{J}_{t^-} = i$ with $i = 1, 2$ and the current inventory level X^π_t level reaches b, it is mandatory to switch off the production. Hence, this time t should coincide with the next switching time T_k for some k and $\mathcal{J}_t = J_k = 0$. Afterwards, $X^\pi_t = b$ for $t \in [T_k, T_{k+1})$, and T_{k+1} would be the time of the arrival of the next costumer demand and J_{k+1} would be either 1 or 2.

If the initial phase is $i \in \{1, 2\}$, given an initial inventory level $x \in [l, b)$, and an admissible production strategy $\pi = (T_k, J_k)_{k\ge0} \in \Pi_{x,i}$, the associated cost function is given by,

$$\begin{aligned} V^\pi_i(x) \;=\; & \mathbb{E}\left[\int_0^\infty e^{-qt} h_{\mathcal{J}t}(X^\pi_t)dt\right] + \mathbb{E}\left[\sum\nolimits_{k=0}^\infty e^{-qT_{k+1}} K_{J_k,J_{k+1}}\right] \\ & + \mathbb{E}\left[\sum\nolimits_{n=1}^\infty e^{-q\tau_n} 1_{\left\{X^\pi_{\tau_n^-} - l < Y_n\right\}}\, p\left(Y_n - X^\pi_{\tau_n^-} + l\right)\right]. \end{aligned}$$

We define the optimal cost functions for $i = 1, 2$ as

$$V_i(x) = \inf_{\pi\in\Pi_{x,i}} V^\pi_i(x) \tag{2.5}$$

for $x \in [l, b)$.

Given an initial inventory level b and an admissible inventory strategy $\pi = (T_k, J_k)_{k\geq 0} \in \Pi_{b,0}$ the cost value of this strategy is given by

$$\begin{array}{rcl} V_0^\pi(b) & = & \mathbb{E}\left[\int_0^\infty e^{-qt} h_{\mathcal{J}t}(X_t^\pi)dt\right] + \mathbb{E}\left[\sum_{k=0}^\infty e^{-qT_{k+1}} K_{J_k,J_{k+1}}\right] \\ & & +\mathbb{E}\left[\sum_{n=1}^\infty e^{-q\tau_n} 1_{\left\{X_{\tau_n^-}^\pi - Y_n < l\right\}}\, p\left(Y_n - X_{\tau_n^-}^\pi + l\right)\right]. \end{array}$$

In this case, the optimal value for inventory level b is given by

$$V_0(b) = \inf_{\pi \in \Pi_{b,0}} V_0^\pi(b). \tag{2.6}$$

3. Basic Properties

In this section we study the existence and regularity of the optimal cost functions. Let us start proving that they are well defined.

Proposition 3.1. *$V_0(b)$ is finite and the optimal cost functions V_i are bounded in $[l, b)$ for $i = 1, 2$. We call $\overline{V}_i$ the positive upper bounds of the functions V_i for $i = 1, 2$.*

Proof.

Take $i \in \{1, 2\}$, $x \in [l, b)$ and the admissible production strategy $\pi = (T_k, J_k)_{k\geq 1} \in \Pi_{x,i}$ that only switch off from phase i to phase 0 when the current inventory level is b and remain in phase i otherwise. Let us call

$$\overline{h} = \max\left\{\sup_{x\in[l,b]} h_1(x), \sup_{x\in[l,b]} h_2(x), h_0(b)\right\}. \tag{3.1}$$

Then, we have

$$\mathbb{E}\left[\int_0^\infty e^{-qt} h_{\mathcal{J}t}(X_t^\pi)dt\right] \leq \frac{\overline{h}}{q}. \tag{3.2}$$

Moreover,

$$\begin{array}{l} \mathbb{E}\left[\sum_{n=1}^{N_t} e^{-q\tau_n} 1_{\left\{X_{\tau_n^-}^\pi < Y_n + l\right\}}\, p\left(Y_n - X_{\tau_n^-}^\pi + l\right)\right] \\ \leq \mathbb{E}\left[\sum_{n=1}^\infty e^{-q\tau_n} p(Y_n)\right] = \mathbb{E}\left[\sum_{n=1}^\infty e^{-q\tau_n}\right] \mathbb{E}[p(Y_1)], \end{array} \tag{3.3}$$

and so it is finite from (2.1). Finally,

(3.4)

$$\begin{array}{rcl} \mathbb{E}\left[\sum_{k=1}^\infty e^{-qT_k} K_{J_{k-1},J_k}\right] & \leq & \mathbb{E}\left[\sum_{k=1}^\infty 1_{\mathcal{J}_{T_k^-}=i} 1_{\mathcal{J}_{T_k}=0} e^{-qT_k} K_{i,0}\right] + \mathbb{E}\left[\sum_{k=1}^\infty 1_{\mathcal{J}_{T_k^-}=0} 1_{\mathcal{J}_{T_k}=i} e^{-qT_k} K_{0,i}\right] \\ & \leq & K_{i,0} + (K_{i,0} + K_{0,i})\, \mathbb{E}\left[\sum_{k=1}^\infty e^{-q\tau_k}\right] \\ & \leq & K_{i,0} + (K_{i,0} + K_{0,i})\, \lambda/q \end{array}$$

so from (3.2), (3.3) and (3.4), the function V_i is bounded in $[l, b)$. With a similar proof it can be shown that V_0 is finite and so we have the result.■

Proposition 3.2. *The optimal cost functions V_i are Lipschitz for $i = 1, 2$ in $[l, b)$.*

Proof.

Given initial inventory level $x \in [l, b)$ and initial phase $i = 1, 2$, take $\delta \in (0, b - x]$ and consider an admissible strategy $\pi_{x+\delta} \in \Pi_{x+\delta,i}$ such that $V_i^{\pi_{x+\delta}}(x + \delta) \leq V_i(x + \delta) + \varepsilon$, where $0 < \varepsilon < \delta$. Let us now define the admissible strategy $\pi_x \in \Pi_{x,i}$ as follows: stay in phase i until the controlled inventory level $X_t^{\pi_x}$ reaches $x + \delta$ and then follow $\pi_{x+\delta} \in \Pi_{x+\delta,i}$. Then, from (3.1) and Proposition 3.1, we get

$$
\begin{array}{rcl}
V_i(x) & \leq & V_i^{\pi_x}(x) \\
& \leq & \int_0^{\frac{\delta}{\sigma_i}} e^{-qt} h_{\mathcal{J}t}(x+\sigma_i t)dt + \mathbb{P}[\tau_1 > \frac{\delta}{\sigma_i}] e^{-q\frac{\delta}{\sigma_i}} V_i^{\pi_{x+\delta}}(x+\delta) \\
& & +\mathbb{P}\left[\tau_1 \leq \frac{\delta}{\sigma_i}\right] \overline{V}_i \\
& \leq & \overline{h}\frac{\delta}{\sigma_i} + e^{-(\lambda+q)\frac{\delta}{\sigma_i}} \left(V_i(x+\delta)+\varepsilon\right) + (1-e^{-\lambda\frac{\delta}{\sigma_i}})\overline{V}_i.
\end{array}
$$

Hence, we have

$$
\begin{array}{rcl}
V_i(x) - V_i(x+\delta) & \leq & \overline{h}\frac{\delta}{\sigma_i} + e^{-(\lambda+q)\frac{\delta}{\sigma_i}} \left(V_i(x+\delta)+\varepsilon\right) - V_i(x+\delta) + (1-e^{-\lambda\frac{\delta}{\sigma_i}})\overline{V}_i \\
& \leq & \overline{h}\frac{\delta}{\sigma_i} + \varepsilon + \lambda\frac{\delta}{\sigma_i}\overline{V}_i.
\end{array}
$$

So, taking

$$
m_i^1 := \frac{\overline{h}}{\sigma_i} + 1 + \frac{\lambda}{\sigma_i}\overline{V}_i,
$$

we obtain

$$
V_i(x) - V_i(x+\delta) \leq m_i^1 \delta. \tag{3.5}
$$

Let us prove now that there exists $m_i^2 > 0$ such that,

$$
V_i(x+\delta) - V_i(x) \leq m_i^2 \delta. \tag{3.6}
$$

We start showing that there exists m such that,

$$
V_i(y) - V_i(l) \leq m\delta \tag{3.7}
$$

for all $y \in [l, l+\delta]$. Given $\varepsilon > 0$ and an initial inventory level l, consider the strategy $\pi_l \in \Pi_{l,i}$ for $i = 1, 2$ such that $V_i^{\pi_l}(l) \leq V_i(l) + \varepsilon$ and call $X_t^{\pi_l}$ the associated process with initial inventory level l. Take also a strategy $\pi_b \in \Pi_{b,0}$ such that $V_0^{\pi_b}(b) \leq V_0(b) + \varepsilon$.

Let us define the admissible strategy $\pi_y \in \Pi_{y,i}$ for initial inventory level $y \in [l, l+\delta]$ as:

- For $0 \leq t \leq T$, follow π_l (and so the associated controlled processes $X_t^{\pi_y} = X_t^{\pi_l} + (y-l)$ for $t < T$), where
$$
T := \min\{t : X_t^{\pi_y} = b \text{ or } \check{X}_t^{\pi_y} - (y-l) = \check{X}_t^{\pi_l} < l\}.
$$
- If $X_T^{\pi_y} = b$, follow π_b for $t \geq T$.
- If $\check{X}_T^{\pi_y} < l$ (and so $X_T^{\pi_y} = X_T^{\pi_l} = l$), follow π_l for $t \geq T$.
- If $l \leq X_T^{\pi_y} < y$ (and so $X_T^{\pi_l} = l$ and $X_T^{\pi_y} = \check{X}_T^{\pi_y}$), also follow the strategy π_l for $t \geq T$.

Given any stopping time τ, let us define $\widehat{V}_i^{\pi_y}(y,\tau)$ as the expected discounted cost of the strategy before τ and $\widetilde{V}_i^{\pi_y}(y,\tau)$ as the expected discounted cost of the strategy after τ. Thus,

$$
\begin{array}{cl}
\multicolumn{2}{l}{V_i(y) - V_i(l) - \varepsilon} \\
\leq & V_i^{\pi_y}(y) - V_i^{\pi_l}(l) \\
\leq & \mathbb{E}\left[\int_0^T e^{-qt}\left(h_{\mathcal{J}_t}(X_t^{\pi_l} + (y-l)) - h_{\mathcal{J}_t}(X_t^{\pi_l})\right)dt\right] + \\
& +\mathbb{E}\left[1_{X_T^{\pi_y}=b}\left(e^{-qT}\left(K_{\mathcal{J}_T,0} + V_0(b) + \varepsilon\right) - \widetilde{V}_i^{\pi_l}(l,T)\right)\right] \\
& +\mathbb{E}\left[1_{\check{X}_T^{\pi_y}<l}\, e^{-qT}\left(V_{\mathcal{J}_T}(l) + \varepsilon + p(l - \check{X}_T^{\pi_y}) - (V_{\mathcal{J}_T}(l) + p(l - \check{X}_T^{\pi_y} + y - l))\right)\right] \\
& +\mathbb{E}\left[1_{\{l \leq X_T^{\pi_y} < y\}} e^{-qT}\left(V_{\mathcal{J}_T}^{\pi_{X_T^{\pi_y}}}(X_T^{\pi_y}) - V_{\mathcal{J}_T}^{\pi_l}(l) + 2\varepsilon\right)\right].
\end{array}
$$

Let n_D be the sum of the numbers of discontinuities of h_1 and h_2. Note that between two customer demands, the inventory level $X_t^{\pi_y}$ goes through at most n_D points of discontinuities of $h_{\mathcal{J}_t}$. Hence, calling $\tau_0 = 0$, we have

$$\text{(3.8)}\qquad \mathbb{E}\left[\int_0^T e^{-qt}\left(h_{\mathcal{J}t}(X_t^{\pi_l}+(y-l))-h_{\mathcal{J}t}(X_t^{\pi_l})\right)dt\right]\leq\left(\frac{m_h}{q}+n_D\frac{\overline{h}}{\sigma_2}\left(1+\frac{\lambda}{q}\right)\right)\delta.$$

Let us call $\widetilde{T}:=\inf\{t:X_t^{\pi_l}=b\}$ and $\widetilde{\tau}$ the time of the first customer demand after T; we have that $\mathbb{P}\left[\widetilde{T}>\widetilde{\tau}\right]\leq 1-e^{-\lambda\frac{\delta}{\sigma_2}}$ and so

$$\text{(3.9)}\qquad \begin{array}{cl} & \mathbb{E}\left[1_{X_T^{\pi_y}=b}\left(e^{-qT}\left(K_{\mathcal{J}_T,0}+V_0(b)\right)-\widetilde{V}_i^{\pi_l}(l,T)\right)\right] \\ \leq & \left(1-e^{-\lambda\frac{\delta}{\sigma_2}}\right)\left(V_0(b)+(K_{1,0}\vee K_{2,0})\right) \\ & +\mathbb{E}\left[1_{X_T^{\pi_y}=b}1_{\widetilde{T}<\widetilde{\tau}}e^{-qT}\left(K_{\mathcal{J}_T,0}+V_0(b)\right)-\widetilde{V}_i^{\pi_l}(l,T)\right] \\ \leq & \lambda\frac{\delta}{\sigma_2}\left(V_0(b)+\max\{K_{1,0},K_{2,0}\}\right) \\ & +\mathbb{E}\left[1_{X_T^{\pi_y}=b}1_{\widetilde{T}<\widetilde{\tau}}e^{-qT}\left(K_{\mathcal{J}_T,0}+V_0(b)\right)-\widetilde{V}_i^{\pi_l}(l,T)\right]. \end{array}$$

Let Δ be the length of time after T in which the process $X_t^{\pi_l}$ reaches b in the event of no arrivals of demands. In this case, we have

$$X_{T+\Delta}^{\pi_l}=b-(y-l)+\int_T^{T+\Delta}e^{-qs}\sigma_{\mathcal{J}_T}ds=b$$

and so $\frac{\delta}{\sigma_1}\leq\Delta\leq\frac{\delta}{\sigma_2}$. Hence, from (2.2),

$$\begin{array}{cl} & \mathbb{E}\left[1_{\{X_T^{\pi_y}=b\}}1_{\{\widetilde{T}<\widetilde{\tau}\}}\widetilde{V}_i^{\pi_l}(l,T)\right] \\ \geq & \mathbb{P}\left[\text{no demands in } t\in\left[T,T+\frac{\delta}{\sigma_2}\right]\right]\ \mathbb{E}\left[e^{-q(T+\Delta)}\left(K_{\mathcal{J}_T,0}+V_0(b)\right)\right] \\ \geq & e^{-(q+\lambda)\frac{\delta}{\sigma_2}}\mathbb{E}\left[e^{-qT}\left(K_{\mathcal{J}_T,0}+V_0(b)\right)\right]. \end{array}$$

Therefore,

$$\text{(3.10)}\qquad \begin{array}{cl} & \mathbb{E}\left[1_{\{X_T^{\pi_y}=b\}}1_{\{\widetilde{T}<\tau_1\}}e^{-qT}\left(K_{\mathcal{J}_T,0}+V_0(b)\right)-\widetilde{V}_i^{\pi_l}(l,T)\right] \\ \leq & \left(1-e^{-(q+\lambda)\frac{\delta}{\sigma_2}}\right)\left(K_{1,0}\vee K_{2,0}+V_0(b)\right) \\ \leq & \frac{q+\lambda}{\sigma_2}\left(K_{1,0}\vee K_{2,0}+V_0(b)\right)\delta. \end{array}$$

Since the penalty function p is non-decreasing, we also have,

$$\text{(3.11)}\qquad \mathbb{E}\left[1_{\left\{\overset{\vee}{X}{}_T^{\pi_y}<l\right\}}e^{-qT}\left(p(l-\overset{\vee}{X}{}_T^{\pi_y})-p(l-\overset{\vee}{X}{}_T^{\pi_y}+y-l)\right)\right]\leq 0.$$

Finally, since the event $l\leq X_T^{\pi_y}<y$ coincides with the arrival of a customer demand,

$$\text{(3.12)}\qquad \begin{array}{cl} & \mathbb{E}\left[1_{\{l\leq X_T^{\pi_y}<y\}}e^{-qT}\left(V_{\mathcal{J}_T}^{\pi_{X_T^{\pi_y}}}(X_T^{\pi_y})-V_{\mathcal{J}_T}(l)\right)\right] \\ = & \mathbb{E}\left[1_{\{l\leq X_T^{\pi_y}<y\}}1_{\{T=\tau_k \text{ for some } k\}}e^{-qT}\left(V_{\mathcal{J}_T}^{\pi_{X_T^{\pi_y}}}(X_T^{\pi_y})-V_{\mathcal{J}_T}(l)\right)\right] \\ \leq & \mathbb{E}\left[e^{-q\tau_1}\max_{z\in[l,y]}\left(V_{\mathcal{J}_T}^{\pi_z}(z)-V_{\mathcal{J}_T}(l)\right)\right] \\ \leq & \frac{\lambda}{q+\lambda}\max_{z\in[l,l+\delta]}\left(V_{\mathcal{J}_T}^{\pi_z}(z)-V_{\mathcal{J}_T}(l)\right). \end{array}$$

Hence, from (3.9), (3.10), (3.11) and (3.12), there exists $\overline{m}$ large enough such that

$$\frac{q}{q+\lambda}\max_{z\in[l,l+\delta]}\left(V_i^{\pi_y}(z)-V_i^{\pi_l}(l)\right)\leq\overline{m}\delta.$$

So, we obtain (3.7) with $m=\overline{m}\,(q+\lambda)\,/q$. The argument to show (3.6) is analogous. ■

4. Hamilton Jacobi Bellman equations

From the definitions (2.5) and (2.6), we can obtain recursive equations relating the optimal cost $V_0(b)$ and the optimal cost functions V_i for $i = 1, 2$; these recursive equations will be used to find the Hamilton-Jacobi-Bellman equations of the optimization problem.

It follows immediately from (2.6) that

$$
\begin{aligned}
V_0(b) &= \mathbb{E}\left[\left(\int_0^{\tau_1} e^{-qs} h_0(b)ds + 1_{\{Y_1 \leq b-l\}} e^{-q\tau_1} \overline{V}(b - Y_1)\right)\right] \\
&\quad + \mathbb{E}\left[\left(1_{\{Y_1 > b-l\}} e^{-q\tau_1} \left(p(Y_1 - b + l) + \overline{V}(l)\right)\right)\right] \\
&= \tfrac{1}{q+\lambda} h_0(b) + \tfrac{\lambda}{q+\lambda} \int_0^{b-l} \overline{V}(b-\alpha) dF(\alpha) \\
&\quad + \tfrac{\lambda}{q+\lambda} \left(\int_{b-l}^{\infty} p(\alpha - b + l) dF(\alpha) + \overline{V}(l)(1 - F(b-l)) \right),
\end{aligned} \tag{4.1}
$$

where

$$
\overline{V}(x) = \min\{K_{01} + V_1(x), K_{02} + V_2(x)\}. \tag{4.2}
$$

For $x \in [l, b)$, let us define

$$
t_x^i := \min\{t : x + \sigma_i t = b\} = \frac{b - x}{\sigma_i}.
$$

Take $\{i, j\} = \{1, 2\}$ and consider any stopping time $T_1 \geq 0$ and $0 < h < t_x^i$. Define $\tau = \tau_1 \wedge T_1 \wedge h$ and

$$
\begin{aligned}
P_i(x, T_1, h) &= \mathbb{E}\left[1_{\{\tau = h < \tau_1 \wedge T_1\}} \left(\int_0^h e^{-qs} h_i(x + \sigma_i s) ds + V_i(x + \sigma_i h) e^{-qh} \right)\right] + \\
&\quad + \mathbb{E}\left[1_{\{\tau = \tau_1 < T_1 \wedge h\}} \left(\int_0^{\tau_1} e^{-qs} h_i(x + \sigma_i s) ds \right)\right] \\
&\quad + \mathbb{E}\left[1_{\{\tau = \tau_1 < T_1 \wedge h\}} 1_{\{Y_1 \leq x + \sigma_i \tau_1 - l\}} e^{-q\tau_1} V_i(x + \sigma_i \tau_1 - Y_1)\right] \\
&\quad + \mathbb{E}\left[1_{\{\tau = \tau_1 < T_1 \wedge h\}} 1_{\left\{Y_1^i > x + \sigma_i \tau_1 - l\right\}} e^{-q\tau_1} \left(p\left(Y_1 - \left(x + \sigma_i \tau_1^i - l\right)\right) + V_i(l) \right)\right] \\
&\quad + \mathbb{E}\left[1_{\{\tau = T_1 < \tau_1 \wedge h\}} \left(\int_0^{T_1} e^{-qs} h_i(x + \sigma_i s) ds + \left(V_j(x + \sigma_i T_1) + K_{i,j}\right) e^{-qT_1} \right)\right].
\end{aligned}
$$

We obtain the following recursive equations

$$
V_i(x) = \inf_{T_1 \geq 0} P_i(x, T_1, h). \tag{4.3}
$$

Let us define the operators,

$$
\begin{aligned}
\mathcal{L}_i(V_i)(x) &:= \sigma_i V_i'(x) - (\lambda + q) V_i(x) + \lambda \int_0^{x-l} V_i(x - \alpha) dF(\alpha) + \lambda \int_{x-l}^{\infty} p(\alpha - x + l) dF(\alpha) \\
&\quad + \lambda V_i(l)(1 - F(x - l)) + h_i(x)
\end{aligned} \tag{4.4}
$$

for $i = 1, 2$. Then, the Hamilton-Jacobi-Bellman equations for V_i, are

$$
\min\{\mathcal{L}_i(V_i)(x), V_j(x) + K_{ij} - V_i(x)\} = 0, \tag{4.5}
$$

for $x \in [l, b)$, $\{i, j\} = \{1, 2\}$. Also, defining

$$
\begin{aligned}
\mathcal{L}_0(V_0)(b) &:= -(q + \lambda) V_0(b) + \lambda \left(\int_0^{b-l} \overline{V}(b - \alpha) dF(\alpha) + \int_{b-l}^{\infty} p(\alpha - b + l) dF(\alpha) \right) \\
&\quad + \lambda \overline{V}(l)(1 - F(b - l) + h_0(b),
\end{aligned} \tag{4.6}
$$

we obtain from (4.1), that

$$
\mathcal{L}_0(V_0)(b) = 0. \tag{4.7}
$$

Definition 4.1. *A function $\underline{u}_i : [l, b] \to \mathbf{R}$ is a viscosity subsolution of (4.5) at $x \in [l, b)$ for $\{i, j\} = \{1, 2\}$ if it is Lipschitz and any continuously differentiable function $\psi_i : [l, b] \to \mathbf{R}$ with $\psi_i(x) = \underline{u}_i(x)$ such that $\underline{u}_i - \psi_i$ reaches the minimum at x satisfies*

$$
\min\{\mathcal{L}_i(\psi_i)(x), V_j(x) + K_{ij} - \underline{u}_i(x)\} \leq 0.
$$

A function $\overline{u}_i : [l,b] \to \mathbf{R}$ is a viscosity supersolution of (4.5) at $x \in [l,b)$ for $\{i,j\} = \{1,2\}$ if it is Lipschitz and any continuously differentiable function $\varphi_i : [l,b] \to \mathbf{R}$ with $\varphi_i(x) = \overline{u}_i(x)$ and such that $\overline{u}_i - \varphi_i$ reaches the maximum at x satisfies

$$\min\{\mathcal{L}_i(\varphi_i)(x), V_j(x) + K_{ij} - \overline{u}_i(x)\} \geq 0.$$

The functions ψ_i and φ_i are called test-functions for subsolution and supersolution respectively. If a function u_i is both a subsolution and a supersolution at x it is called a viscosity solution of (4.5) at x.

Crandall and Lions [7] introduced the concept of viscosity solutions for first-order Hamilton-Jacobi equations. It is the standard tool for studying HJB equations, see for instance Fleming and Soner [12].

Proposition 4.2. *The optimal cost functions V_i satisfy (4.5) in a viscosity sense, for $x \in [l,b)$ and $i = 1,2$.*

Proof.

Consider $\{i,j\} = \{1,2\}$, taking $x \in [l,b)$ and $T_1 = 0$ in (4.3), it follows that $V_j(x) + K_{ij} - V_i(x) \geq 0$. Take now $0 < h < T_1$ and $h < t_x^i$. From (4.3) and using that φ_i is a test-functions for supersolution

$$\begin{array}{rcl} \varphi_i(x) = V_i(x) & \leq & \mathbb{E}\left[1_{h<\tau_1}\left(\int_0^h e^{-qs}h_i(x+\sigma_i s)ds + \varphi_i(x+\sigma_i h)e^{-qh}\right)\right] \\ & & +\mathbb{E}\left[1_{\tau_1<h}\left(\int_0^{\tau_1} e^{-qs}h_i(x+\sigma_i s)ds\right)\right] \\ & & +\mathbb{E}\left[1_{\tau_1<h}1_{\{Y_1\leq x+\sigma_i\tau_1-l\}}e^{-q\tau_1}\varphi_i(x+\sigma_i\tau_1-Y_1)\right] \\ & & +\mathbb{E}\left[1_{\tau_1<h}1_{\{Y_1^i>x+\sigma_i\tau_1-l\}}e^{-q\tau_1}\left(p\left(Y_1-\left(x+\sigma_i\tau_1^i-l\right)\right)+\varphi_i(l)\right)\right]. \end{array}$$

Hence,

$$\begin{array}{rcl} 0 & \leq & \mathbb{E}\left[1_{h<\tau_1}\left(\int_0^h e^{-qs}h_i(x+\sigma_i s)ds + \varphi_i(x+\sigma_i h)e^{-qh}\right)\right] - \varphi_i(x) \\ & & +\mathbb{E}\left[1_{\tau_1<h}\left(\int_0^{\tau_1} e^{-qs}h_i(x+\sigma_i s)ds\right)\right] \\ & & +\mathbb{E}\left[1_{\tau_1<h}1_{\{Y_1\leq x+\sigma_i\tau_1-l\}}e^{-q\tau_1}\varphi_i(x+\sigma_i\tau_1-Y_1)\right] \\ & & +\mathbb{E}\left[1_{\tau_1<h}1_{\{Y_1^i>x+\sigma_i\tau_1-l\}}e^{-q\tau_1}\left(p\left(Y_1-\left(x+\sigma_i\tau_1^i-l\right)\right)+\varphi_i(l)\right)\right], \end{array}$$

and so, dividing by h and taking $h \to 0^+$, we obtain $\mathcal{L}_i(\varphi_i)(x) \geq 0$. Hence V_i is a viscosity supersolution of (4.5) at x.

Let us prove now that V_i is a viscosity subsolution of (4.5) at any $x \in (l,b)$ for $i = 1,2$. It is enough to consider the case $V_j(x) + K_{ij} - V_i(x) > 0$. Arguing by contradiction, we assume that V_i is not a subsolution of (4.5) at x. We can find, as in Proposition 3.1 in Azcue and Muler [2], values $\varepsilon > 0$, $h \in (0,(x-l)/2 \wedge (b-x)/2)$ and a continuously differentiable function $\psi_i \geq V_i$ in $[0,y+h]$ with $\psi_i(x) = V_i(x)$ such that

$$\begin{array}{l} V_j(y) + K_{ij} - V_i(y) \geq 0 \text{ for } y \in [l,b), \\ \mathcal{L}_i(\psi_i)(y) \geq 2q\varepsilon \text{ for } y \in [x-h,x+h], \\ V_i(y) \geq \psi_i(y) + 3\varepsilon \text{ for } y \in [l,x-h] \cup \{x+h\}, \end{array}$$

and also

$$V_j(y) + K_{ij} - V_i(y) > 2\varepsilon \text{ for } y \in [x-h,x+h].$$

Since ψ_i is continuously differentiable we can find a positive constant C such that $\mathcal{L}_i(\psi_i)(y) \leq C$ for all $y \in [l,b)$.

Let us take any admissible production strategy $\pi = (T_k,J_k)_{k\geq 1} \in \Pi_{x,i}$, consider the uncontrolled inventory process X_t^π defined in (2.3), and define the stopping times

$$\overline{\tau} = \inf\{t > 0 : X_t^\pi \geq x+h\}, \underline{\tau} = \inf\{t > 0 : X_t^\pi \leq x-h\},$$

and $\tau^* = T_1 \wedge \underline{\tau} \wedge \overline{\tau}$. We get that if $\tau^* = T_1$ and $X_{\tau^*}^\pi \in (x-h,x+h)$ then

$$V_j(X_{T_1}^\pi) + K_{ij} \geq V_i(X_{T_1}^\pi) + 2\varepsilon \geq \psi_i(X_{T_1}^\pi) + 2\varepsilon,$$

and in the case that either $\tau^* < T_1$ or $\tau^* = T_1$ and $X_{\tau^*}^\pi \notin (x-h,x+h)$ we have that

$$V_i(X_{\tau^*}^\pi) \geq \psi_i(X_{\tau^*}^\pi) + 2\varepsilon.$$

Since the function $e^{-qt}\psi_i(x)$ is continuously differentiable, using the expression (2.3) and the change of variables formula for finite variation processes (see Protter [20]), we can write

$$(4.8)\qquad \begin{array}{rl} \psi_i(X^\pi_{\tau^*})e^{-q\tau^*} - \psi_i(x) = & \int_0^{\tau^*} \psi_i'(X^\pi_{s^-})e^{-qs}\sigma_i ds - q\int_0^{\tau^*} \psi_i(X^\pi_{s^-})e^{-qs}ds \\ + & \sum\limits_{X_{s^-}\neq X_s, s\leq \tau^*} \left(\psi_i(X^\pi_{s^-} - \Delta X_s) - \psi_i(X^\pi_{s^-})\right)e^{-qs}, \end{array}$$

where $\Delta X_s = X_s - X_{s^-}$.

On the other hand, $X_s \neq X_{s^-}$ only at the arrival of a demand, so

$$(4.9)\qquad \begin{array}{rl} M_t = & \sum\limits_{X_{s^-}\neq X_s, s\leq t} \left(\psi_i(X^\pi_{s^-} - \Delta X_s) - \psi_i(X^\pi_{s^-})\right)e^{-qs} \\ & -\lambda\int_0^t e^{-qs}\int_0^\infty \left(\psi_i(X^\pi_{s^-} - \alpha) - \psi_i(X^\pi_{s^-})\right)dF(\alpha)ds \end{array}$$

is a martingale with zero-expectation, here we extend the definition of ψ_i for $y < l$ as $\psi_i(y) = p(l-y) + \psi_i(l)$. Therefore, we can combine (4.8) and (4.9) to obtain

$$(4.10)\qquad \psi_i(X^\pi_{\tau^*})e^{-q\tau^*} - \psi_i(x) = \int_0^{\tau^*}\mathcal{L}_i(\psi_i)(X^\pi_{s^-})e^{-qs}ds + M_{\tau^*} - \int_0^{\tau^*} h_i(X^\pi_{s^-})e^{-qs}ds.$$

In the case that $\tau^* = T_1$ and $X^\pi_{T_1} \in (x-h, x+h)$, we have from (4.10) that

$$\left(V_j(X^\pi_{T_1}) + K_{ij}\right)e^{-q\tau^*} + \int_0^{\tau^*} h_i(X^\pi_{s^-})e^{-qs}ds \geq V_i(x) + 2\varepsilon + M_{\tau^*}.$$

In the case that either $\tau^* < T_1$ or $\tau^* = T_1$ and $X^\pi_{\tau^*} \notin (x-h, x+h)$, we get

$$(4.11)\qquad \begin{array}{rcl} (V_i(X^\pi_{\tau^*}) - 2\varepsilon)e^{-q\tau^*} - V_i(x) & \geq & \psi_i(X^\pi_{\tau^*})e^{-q\tau^*} - \psi_i(x) \\ & = & \int_0^{\tau^*}\mathcal{L}_i(\psi_i)(X^\pi_{s^-})e^{-qs}ds + M_{\tau^*} - \int_0^{\tau^*} h_i(X^\pi_{s^-})e^{-qs}ds. \\ & \geq & \int_0^{\tau^*} 2q\varepsilon e^{-qs}ds + M_{\tau^*} - \int_0^{\tau^*} h_i(X^\pi_{s^-})e^{-qs}ds. \\ & = & 2\varepsilon(1 - e^{-q\tau^*}) + M_{\tau^*} - \int_0^{\tau^*} h_i(X^\pi_{s^-})e^{-qs}ds. \end{array}$$

and so, by (4.10) and (4.11),

$$e^{-q\tau^*}V_i(X^\pi_{\tau^*}) + \int_0^{\tau^*} h_i(X^\pi_{s^-})e^{-qs}ds \geq V_i(x) + \varepsilon + M^1_{\tau^*}.$$

Finally, we obtain that $V_i^\pi(x) \geq V_i(x) + 2\varepsilon$, and this contradicts the definition of V_i. ■

In the following proposition, we prove that the optimal cost functions are the largest viscosity supersolutions of their corresponding HJB equations with suitable boundary conditions.

Proposition 4.3. *Fix $x \in [l,b)$ and $j = 1,2$ or $x = b$ and $j = 0$. Let $\overline{u}_1$ and $\overline{u}_2$ be non-negative viscosity supersolution of the corresponding HJB equation (4.5) in $[l,b)$ and consider any admissible strategy $\pi = (T_k, J_k)_{k\geq 0} \in \Pi_{x,j}$. Defining*

$$\overline{u}(x) = \min\{K_{01} + \overline{u}_1(x), K_{02} + \overline{u}_2(x)\}$$

and since $\mathcal{L}_0(\overline{u}_0)(b) = 0$,

$$\overline{u}_0(b) = \frac{\lambda}{q+\lambda}\left(\int_0^{b-l}\overline{u}(b-\alpha)dF(\alpha) + \int_{b-l}^\infty \left(p(\alpha - b + l) + \overline{u}(l)\right)dF(\alpha)\right) + \frac{h_0(b)}{q+\lambda}.$$

If we assume that

$$\overline{u}_1(b) \leq \overline{u}_0(b) + K_{10},\ \overline{u}_2(b) \leq \overline{u}_0(b) + K_{20},$$

then $\overline{u}_j(x) \leq V_j^\pi(x)$ for $j = 1,2$ and $\overline{u}_0(b) \leq V_0^\pi(b)$.

Proof.

Consider $\pi \in \Pi_{x,j}$. Let us extend $\overline{u}_1$ and $\overline{u}_2$ as $\overline{u}_i(x) = \overline{u}_i(l)$ and $\overline{u}_0(x) = \overline{u}_0(l)$ for $x < l$. Consider the controlled risk process X_t^π starting at x and the function $\mathcal{J}_t$ defined in (2.4). Since $\overline{u}_i$ is Lipschitz for $i = 1,2$, we obtain that the function $t \to e^{-qt}\ \overline{u}_{\mathcal{J}_t}(X_t^\pi)$ is absolutely continuous in between the stopping times $\{0\} \cup \{\tau_n : n \geq 1\} \cup \{T_k : k \geq 1\}$. So, taking

$$m_t := \max\{k : T_k \leq t\},$$

we have
(4.12)

$$\begin{aligned}&\overline{u}_{\mathcal{J}_t}(X_t^\pi)e^{-qt}-\overline{u}_j(x)\\ =&\ \sum_{k=0}^{m_t-1}\left(\overline{u}_{J_{k+1}}(X_{T_{k+1}}^\pi)e^{-qT_{k+1}}-\overline{u}_{J_k}(X_{T_k}^\pi)e^{-qT_k}\right)+\left(\overline{u}_{J_{m_t}}(X_t^\pi)e^{-qt}-\overline{u}_{J_{m_t}}(X_{T_{m_t}}^\pi)e^{-qT_{m_t}}\right).\end{aligned}$$

Let us define

$$\begin{aligned}M^i(z_0,t_0,t) &= \overline{u}_i(Z_t^i)e^{-qt}-\overline{u}_i(z_0)e^{-qt_0}+\sum_{n=N_{t_0}}^{Nt}e^{-q\tau_n}p(l-Z^i_{\tau_n^-}+Y_n)1_{\{Z^i_{\tau_n^-}-Y_n-l<0\}}\\ &\quad-\int_{t_0}^t e^{-qs}\left(\sigma_i\overline{u}_i'(Z_s^i)-(q+\lambda)\overline{u}_i(Z_s^i)+\lambda\int_0^{Z^i_{s^-}-l}\overline{u}_i(Z^i_{s^-}-\alpha)dF(\alpha)\right)ds\\ &\quad-\int_{t_0}^t e^{-qs}\left(\lambda\int_{Z^i_{s^-}-l}^\infty\left(p(\alpha-Z^i_{s^-}+l)+\overline{u}_i(l)\right)dF(\alpha)\right)ds\end{aligned}\tag{4.13}$$

with

$$Z_t^i=z_0+\sigma_i\,(t-t_0)-\sum_{n=N_{t_0}}^{N_t}\min\{Y_n,Z^i_{\tau_n^-}-l\}\text{ for }t\geq t_0\geq 0,$$

it can be seen that $M^i(z_0,t_0,t)$ is a martingale with zero expectation for $t\geq t_0$.

Consider first the case $J_k=i$ and $J_{k+1}=j$ with $i=1,2$, $j=0,1,2$ and $i\neq j$. Since $\overline{u}_i$ is absolutely continuous, the function $t\to\overline{u}_i(X_t^\pi)e^{-qt}$ is also absolutely continuous, between the customer demands. Using an extension of the Dynkin's Formula, we obtain

$$\begin{aligned}&\overline{u}_j(X^\pi_{T_{k+1}})e^{-qT_{k+1}}-\overline{u}_i(X^\pi_{T_k})e^{-qT_k}\\ =&\ \overline{u}_j(X^\pi_{T_{k+1}})e^{-qT_{k+1}}-\overline{u}_i(X^\pi_{T_{k+1}})e^{-qT_{k+1}}+\overline{u}_i(X^\pi_{T_{k+1}})e^{-qT_{k+1}}-\overline{u}_i(X^\pi_{T_k})e^{-qT_k}\\ \geq&\ -K_{ij}e^{-qT_{k+1}}+\overline{u}_i(X^\pi_{T_{k+1}})e^{-qT_{k+1}}-\overline{u}_i(X^\pi_{T_k})e^{-qT_k}\\ =&\ -K_{ij}e^{-qT_{k+1}}+\int_{T_k}^{T_{k+1}}e^{-qs}\mathcal{L}_i(\overline{u}_i)(X_s^\pi)ds\\ &\ -\left(\int_{T_k}^{T_{k+1}}e^{-qs}h_i(X_s^\pi)ds+\sum_{n=N_{T_k}}^{N_{T_{k+1}}}e^{-q\tau_n}p(l-X^\pi_{\tau_n^-}+Y_n)1_{\{X^\pi_{\tau_n^-}-Y_n-l<0\}}\right)\\ &\ +M^i(X^\pi_{T_k},T_k,T_{k+1});\end{aligned}$$

and so, since $\overline{u}_i$ is a supersolution of (4.5), we get that

$$\begin{aligned}&\mathbb{E}\left[\overline{u}_j(X^\pi_{T_{k+1}})e^{-qT_{k+1}}-\overline{u}_i(X^\pi_{T_k})e^{-qT_k}\,\middle|\,\mathcal{F}_{T_k}\right]\\ \geq&\ -\mathbb{E}\left[K_{ij}e^{-qT_{k+1}}+\int_{T_k}^{T_{k+1}}e^{-qs}h_i(X_s^\pi)ds+\sum_{n=N_{T_k}}^{N_{T_{k+1}}}e^{-q\tau_n}p(l-X^\pi_{\tau_n^-}+Y_n)1_{\{X^\pi_{\tau_n^-}-Y_n-l<0\}}\,\middle|\,\mathcal{F}_{T_k}\right].\end{aligned}$$

In the case $J_k=0$ we have $\mathcal{J}_{T_{k+1}}\neq 0$, then $X_s^\pi=b$ in $[T_k,T_{k+1})$, $T_{k+1}=\tau_n$ for some n and so, analogously to the previous case,

$$\begin{aligned}&\overline{u}_{\mathcal{J}_{T_{k+1}}}(X^\pi_{T_{k+1}})e^{-qT_{k+1}}-\overline{u}_0(X^\pi_{T_k})e^{-qT_k}\\ =&\ e^{-qT_k}\left((\overline{u}(b-Y_n)1_{\{b-Y_n-l\geq 0\}}+\overline{u}_{\mathcal{J}_{T_{k+1}}}(l)1_{\{b-Y_n-l<0\}})e^{-q(T_{k+1}-T_k)}-\overline{u}_0(b)\right)-K_{0\mathcal{J}_{T_{k+1}}}e^{-qT_{k+1}}\\ =&\ e^{-qT_k}\left(\overline{u}(b-Y_n)1_{\{b-Y_n-l\geq 0\}}+\overline{u}_{\mathcal{J}_{T_{k+1}}}(l)1_{\{b-Y_n-l<0\}}+p(l-b+Y_n)1_{\{b-Y_n-l<0\}}\right)e^{-q(T_{k+1}-T_k)}\\ &\ -e^{-qT_k}\left(\tfrac{\lambda}{q+\lambda}\left(\int_0^{b-l}\overline{u}(b-\alpha)dF(\alpha)+\int_{b-l}^\infty\left(p(\alpha-b+l)+\overline{u}(l)\right)dF(\alpha)\right)\right)\\ &\ -\left(K_{0\mathcal{J}_{T_{k+1}}}e^{-qT_{k+1}}+\int_{T_k}^{T_{k+1}}e^{-qs}h_0(b)ds+e^{-qT_{k+1}}p(l-b+Y_n)1_{\{b-Y_n-l<0\}}\right)\\ &\ +\int_{T_k}^{T_{k+1}}e^{-qs}h_0(b)ds-\tfrac{\lambda}{q+\lambda}e^{-qT_k}h_0(b),\end{aligned}$$

and, since $T_{k+1}-T_k$ is distributed as $exp(\lambda)$, we obtain that

$$\begin{aligned}0=&\ \mathbb{E}\left[e^{-qT_k}\left(\overline{u}(b-Y_n)1_{\{b-Y_n-l\geq 0\}}+\overline{u}_{\mathcal{J}_{T_{k+1}}}(l)1_{\{b-Y_n-l<0\}}\right)\,\middle|\,\mathcal{F}_{T_k}\right]\\ &+\mathbb{E}\left[e^{-qT_{k+1}}p(l-b+Y_n)1_{\{b-Y_n-l<0\}}\,\middle|\,\mathcal{F}_{T_k}\right]\\ &-\mathbb{E}\left[\tfrac{e^{-qT_k}}{q+\lambda}\lambda\left(\int_0^{b-l}\overline{u}(b-\alpha)dF(\alpha)+\int_{b-l}^\infty\left(p(\alpha-b+l)+\overline{u}(l)\right)dF(\alpha)\right)\,\middle|\,\mathcal{F}_{T_k}\right].\end{aligned}$$

and so

$$\begin{aligned}&\mathbb{E}\left[\overline{u}_{\mathcal{J}_{T_{k+1}}}(X^\pi_{T_{k+1}})e^{-qT_{k+1}}-\overline{u}_0(X^\pi_{T_k})e^{-qT_k}\,\middle|\,\mathcal{F}_{T_k}\right]\\ =&\ -\mathbb{E}\left[K_{0\mathcal{J}_{T_{k+1}}}e^{-qT_{k+1}}+\int_{T_k}^{T_{k+1}}e^{-qs}h_0(b)ds+e^{-qT_{k+1}}p(l-b+Y_n)1_{\{b-Y_n-l<0\}}\,\middle|\,\mathcal{F}_{T_k}\right].\end{aligned}$$

Analogously, we can prove that

$$\begin{aligned}&\mathbb{E}\left[\overline{u}_{J_{m_t}}(X_t^{\pi})e^{-qt}-\overline{u}_{J_{m_t}}(X_{T_{m_t}}^{\pi})e^{-qT_{m_t}}\,\middle|\,\mathcal{F}_{T_{m_t}}\right]\\&\geq -\mathbb{E}\left[\int_{T_{m_t}}^{t}e^{-qs}h_{J_{m_t}}(X_s^{\pi})ds+\sum_{n=N_{T_{m_t}}}^{t}e^{-q\tau_n}p(l-X_{\tau_n^-}^{\pi}+Y_n)1_{\{X_{\tau_n^-}^{\pi}-Y_n-l<0\}}\,\middle|\,\mathcal{F}_{T_{m_t}}\right].\end{aligned}$$

Taking the expected value in (4.12), we obtain

$$\begin{aligned}\mathbb{E}\left[\overline{u}_{\mathcal{J}_t}(X_t^{\pi})e^{-qt}\right]-\overline{u}_j(x)&=\mathbb{E}\left[\sum_{k=0}^{m_t-1}\mathbb{E}\left[\left(\overline{u}_{J_{k+1}}(X_{T_{k+1}}^{\pi})e^{-qT_{k+1}}-\overline{u}_{J_k}(X_{T_k}^{\pi})e^{-qT_k}\right)\middle|\,\mathcal{F}_{T_k}\right]\right]\\&\quad+\mathbb{E}\left[\mathbb{E}\left[\left(\overline{u}_{J_{m_t}}(X_t^{\pi})e^{-qt}-\overline{u}_{J_{m_t}}(X_{T_{m_t}}^{\pi})e^{-qT_{m_t}}\right)\middle|\,\mathcal{F}_{T_{m_t}}\right]\right]\\&\geq -V_j^{\pi}(x)\end{aligned}$$

taking the limit with t going to infinity, and using that $X_t^{\pi}\in[l,b]$ we obtain that $\overline{u}_j(x)\leq V_j^{\pi}(x)$ for $j=1,2$.

Considering instead the controlled risk process X_t^{π} starting at b, we obtain with a similar proof that $\overline{u}_0(b)\leq V_0^{\pi}(b)$. ■

From Propositions 4.2 and 4.3, we obtain the following verification result.

Theorem 4.4. *Consider two families of admissible strategies* $\{\pi_{x,i}\in\Pi_{x,i}:x\in[l,b)\}$ *for* $i=1,2$. *If the functions* $w_i(x):=V_i^{\pi_{x,i}}(x)$ *for* $i=1,2$ *are viscosity supersolutions of the respective HJB equation (4.5) for* $x\in(l,b)$ *and satisfy the boundary conditions*

$$w_1(b)\leq w_0(b)+K_{10},w_2(b)\leq w_0(b)+K_{20},$$

where

$$\begin{aligned}&w_0(b)=\tfrac{\lambda}{q+\lambda}\left(\int_0^{b-l}\overline{w}(b-\alpha)dF(\alpha)+\int_{b-l}^{\infty}\left(p(\alpha-b+l)+\overline{w}(l)\right)dF(\alpha)\right)+\tfrac{h_0(b)}{q+\lambda}\text{ and}\\&\overline{w}(x)=\min\{K_{01}+w_1(x),K_{02}+w_2(x)\}.\end{aligned}$$

Then, $w_0(b)=V_0(b)$ *and* $w_i=V_i$ *for* $i=1,2$.

In the remainder of the section, we show that there exists an optimal production-inventory strategy and it is stationary in the sense that depends only on the phase and the inventory level.

Definition 4.5. *Given two disjoint closed sets* A_{12} *and* A_{21} *in* $[l,b)$ *and a closed set* C_1 *in* $[l,b)$ *with* $A_{21}\subset C_1$ *and* $A_{12}\subset[l,b)-C_1$, *we define the production-inventory band strategy associated to the sets* (A_{12},A_{21},C_1) *as follows:*

1. If the current phase is $i=1$ and the current inventory level is $x\in A_{12}$, change immediately to phase 2, if the current inventory level $x\in[l,b)-A_{12}$ stay in phase 1.

2. If the current phase is $i=2$ and the current inventory level is $x\in A_{21}$, change immediately to phase 1, if the current inventory level $x\in[l,b)-A_{21}$ stay in phase 2.

3. If the current phase is $i=0$ with current inventory level b, then in the event of an arrival of the next customer demand of size Y, switch on the production to phase 1 if $\max\{b-Y,l\}\in C_1$ and switch on the production to phase 2 if $\max\{b-Y,l\}\in[l,b)-C_1$.

4. If the inventory level reaches b, it is mandatory to switch to phase 0.

The sets A_{ij} are called the *switching zone* from the phase i to phase j, and the sets C_1 and $C_2=[l,b)-C_1$ are called the *selection zones* for phases 1 and 2 respectively. Also, the set $[l,b)-(A_{12}\cup A_{21})$ is called the *non-action zone.*

Remark 4.6. *Given the sets* $\mathcal{A}=(A_{12},A_{21},C_1)$, *an initial inventory level* x *and an initial phase* i, *we define and admissible strategy* $\pi_{x,i}^{\mathcal{A}}=(T_k,J_k)_{k\geq0}\in\Pi_{x,i}$ *where* $J_0=i$ *and* T_k *is the* k*-th switching (from regime* J_{k-1} *to* J_k*) given by* (1), (2), (3) *and* (4). *Note that the switching times* T_k *are the times in which the controlled inventory process in* $[l,b]$ *exit the sets* $[l,b)-A_{12}$, $[l,b)-A_{21}$ *and* $\{b\}$. *Let us denote the cost function of this admissible strategies as*

$$W_i^{\mathcal{A}}(x)=V_i^{\pi_{x,i}^{\mathcal{A}}}(x)\text{ for }i=1,2\text{ and }x\in[l,b);\text{ and }W_0^{\mathcal{A}}(b)=V_i^{\pi_{b,0}^{\mathcal{A}}}(b).$$

We can characterize the triple $(W_0^{\mathcal{A}}(b), W_1^{\mathcal{A}}, W_2^{\mathcal{A}})$ as the unique fixed point of a contraction operator: Let $\mathcal{C}[l,b)$ be the set of all the functions $W : [l,b) \to \mathbf{R}$ continuous and bounded and let consider the Banach space

$$\mathcal{B} = \mathbf{R} \times \mathcal{C}[l,b) \times \mathcal{C}[l,b)$$

with norm

$$\|(f_0, f_1, f_2)\| = \max\{|f_0|, \sup_{x\in[l,b)} |f_1(x)|, \sup_{x\in[l,b)} |f_2(x)|\}.$$

We define, the operator $\mathcal{T}^{\mathcal{A}} : \mathcal{B} \to \mathcal{B}$ as

$$\mathcal{T}^{\mathcal{A}}(f_0, f_1, f_2) = \left(\mathcal{T}_0^{\mathcal{A}}(f_1, f_2, f_0), \mathcal{T}_1^{\mathcal{A}}(f_1, f_2, f_0), \mathcal{T}_2^{\mathcal{A}}(f_1, f_2, f_0)\right). \tag{4.14}$$

We define $\mathcal{T}_0^{\mathcal{A}}$ as

$$\begin{array}{rcl} \mathcal{T}_0^{\mathcal{A}}(f_0, f_1, f_2) & := & \mathbb{E}\left[\int_0^{\tau_1} e^{-qs} h_0(b) ds)\right] \\ & & +\mathbb{E}\left[1_{\{Y_1 \le b-l\}} e^{-q\tau_1} \left(\overline{f}(b - Y_1)\right)\right] \\ & & +\mathbb{E}\left[1_{\{Y_1 > b-l\}} e^{-q\tau_1} \left(p(Y_1 - b + l) + \overline{f}(l)\right)\right] \end{array}$$

where

$$\overline{f}(x) \quad := \quad (f_1(x) + K_{01}) 1_{\{x\in C_1^*\}} + (f_2(x) + K_{02}) 1_{\{x\notin C_1^*\}},$$

here (τ_1, Y_1) is the time and size of the first costumer demand. Take the admissible strategy $\pi_{x,i}^{\mathcal{A}} = (T_k, J_k)_{k\geq 0} \in \Pi_{x,i}$ as defined in Definition 4.5 and consider the associated controlled inventory process X_t and the process $\mathcal{J}_t$ defined in (2.4), we define $\mathcal{T}_i^{\mathcal{A}}$ as

$$\begin{array}{rcl} \mathcal{T}_i^{\mathcal{A}}(f_0, f_1, f_2)(x) & = & \mathbb{E}\left[\int_0^{\tau_1} e^{-qs} h_{\mathcal{J}s}(X_s) ds\right] + \\ & & +\mathbb{E}\left[\sum_{k=1}^{\infty} 1_{\{T_k<\tau_1\}} e^{-qT_k} K_{J_{k-1}J_k}\right] \\ & & +\mathbb{E}\left[1_{\left\{X_{\tau_1^-} - Y_1 \ge l\right\}} e^{-q\tau_1} \left(f_{\mathcal{J}\tau_1^-}(X_{\tau_1^-} - Y_1)\right)\right] \\ & & +\mathbb{E}\left[1_{\left\{X_{\tau_1^-} - Y_1 < l\right\}} e^{-q\tau_1} \left(p(l - X_{\tau_1^-} + Y_1) + f_{\mathcal{J}\tau_1^-}(l)\right)\right] \end{array}$$

for $x \in [l,b)$ and $i \in \{1,2\}$. Note that

$$\begin{array}{rcl} \left|\mathcal{T}_i^{\mathcal{A}}(f_0, f_1, f_2) - \mathcal{T}_i^{\mathcal{A}}(g_0, g_1, g_2)\right| & \le & \mathbb{E}\left(e^{-q\tau_1}\right) \|(f_0, f_1, f_2) - (g_0, g_1, g_2)\| \\ & = & \frac{\lambda}{q+\lambda} \|(f_0, f_1, f_2) - (g_0, g_1, g_2)\| \end{array}$$

and so $\mathcal{T}^{\mathcal{A}} : \mathcal{B} \to \mathcal{B}$ is a contraction operator with a unique fixed point. Finally, by the definition of the production-inventory strategy associated to the sets $\mathcal{A} = (A_{12}, A_{21}, C_1)$, it follows immediately that the triple $(W_0^{\mathcal{A}}(b), W_1^{\mathcal{A}}, W_2^{\mathcal{A}})$ is a fixed point of the operator $\mathcal{T}^{\mathcal{A}}$.

In the following theorem we prove that there exists an optimal strategy and that it comes from a production-inventory band strategy as defined in Definition 4.5.

Theorem 4.7. *The optimal strategy of problem (2.5) and (2.6), is the production-inventory strategy associated to the sets $\mathcal{A}^* = (A_{12}^*, A_{21}^*, C_1^*)$ where*

$$\begin{array}{rcl} A_{12}^* & = & \{x \in [l,b) : V_2(x) + K_{12} - V_1(x) = 0\}, \\ A_{21}^* & = & \{x \in [l,b) : V_1(x) + K_{21} - V_2(x) = 0\}, \\ C_1^* & = & \{x \in [l,b) : K_{01} + V_1(x) \le K_{02} + V_2(x)\}. \end{array}$$

Proof.

By Remark 4.6, it is enough to prove that the triple $(V_0(b), V_1, V_2)$ is a fixed point of the operator $\mathcal{T}^{\mathcal{A}^*}$ for the sets $\mathcal{A}^* = (A_{12}^*, A_{21}^*, C_1^*)$. By definition of the sets A_{ij}^* and C_1^*, we obtain immediately that $\mathcal{T}_0^{\mathcal{A}^*}(V_0(b), V_1, V_2) = V_0(b)$. Let us prove now that $\mathcal{T}_i^{\mathcal{A}^*}(V_0(b), V_1, V_2)(x) = V_i(x)$ for $x \in [l,b)$ and $i = 1, 2$. Since $\mathcal{L}_0(V_0)(b) = 0$; and for $\{i,j\} = \{1,2\}$ the functions $t \to V_i(X_t)$ are absolutely continuos, $\mathcal{L}_i(V_i) = 0$ a.e. in $[l,b) - A_{ij}^*$ and $V_j(x) + K_{ij} - V_i(x) = 0$ in A_{ij}^*; we can prove, with arguments similar to the proof of Proposition 4.3, and using the martingales introduced in (4.13) that

$$
\begin{aligned}
\mathcal{T}_i^{\mathcal{A}^*}(V_0,V_1,V_2)(x)-V_i(x) &= \mathbb{E}\left[\int_0^{\tau_1} e^{-qs}h_{\mathcal{J}s}(X_s)ds\right]+\\
&+\mathbb{E}\left[\sum_{k=1}^{\infty}1_{\{T_k<\tau_1\}}e^{-qT_k}K_{J_{k-1}J_k}\right]\\
&+\mathbb{E}\left[1_{\left\{X_{\tau_1^-}-Y_1\geq l\right\}}e^{-q\tau_1}\left(V_{\mathcal{J}\tau_1^-}(X_{\tau_1^-}-Y_1)\right)\right]\\
&+\mathbb{E}\left[1_{\left\{X_{\tau_1^-}-Y_1<l\right\}}e^{-q\tau_1}\left(p(l-X_{\tau_1^-}+Y_1)+V_{\mathcal{J}\tau_1^-}(l)\right)\right]-V_i(x).\\
&= \mathbb{E}\left[\int_0^{\tau_1} e^{-qs}\mathcal{L}_{\mathcal{J}s}(V_{\mathcal{J}s})(X_s)ds\right]\\
&= 0.
\end{aligned}
$$

Hence, $W_0^{\mathcal{A}^*}(b)=V_0(b), W_1^{\mathcal{A}^*}=V_1$ and $W_2^{\mathcal{A}^*}=V_2$.■

5. Finite Band strategies

We define the finite band strategies as the production-inventory band strategies in which the non-action set $[l,b)-(A_{12}\cup A_{21})$ has a finite number of connected components.

Doshi et al. [11] studied the production-inventory band strategies with switching zones $A_{12}=[y_1,b)$ and $A_{21}=[l,y_2]$ and selection zones $C_1=[l,y_2]$ and $C_2=(y_2,b)$ for $l\leq y_2<y_1<b$.

Assuming that the optimal strategy is a finite band strategy, we look for it in the following way;

First step. We find the best *Doshi strategy*, that is we construct the cost functions $(W_0^{\mathcal{A}}(b),W_1^{\mathcal{A}},W_2^{\mathcal{A}})$ for $\mathcal{A}=([y_1,b),[l,y_2],(y_2,b))$; then we minimize the $W_0^{\mathcal{A}}(b)$ among the two variables $l\leq y_2<y_1<b$. We check whether the associated cost functions $W_0^{\mathcal{A}}(b),W_1^{\mathcal{A}}$ and $W_2^{\mathcal{A}}$ of this strategy satisfy the conditions of Theorem 4.4, if they do this is the optimal strategy; if this is not the case, we go to the second step.

Second step. We consider the *band strategies of type one* where the non-action zone has one connected component. Here, the switching zones are of the form $A_{12}=[y_1,b)$ and $A_{21}=[l,y_2]$ and the selection zones are of the form $C_1=[l,y_3]$ and $C_2=(y_3,b)$ for $l\leq y_2\leq y_3<y_1<b$; the non-action zone is (y_2,y_1). Then we minimize $W_0^{\mathcal{A}}(b)$ among the three variables $y_2,\ y_3\ ,\ y_1$. As before, we check whether the associated cost functions $W_0^{\mathcal{A}}(b),W_1^{\mathcal{A}}$ and $W_2^{\mathcal{A}}$ of this strategy satisfy the conditions of Theorem 4.4, if they do this is the optimal strategy; if this is not the case, we go to the third step. Note that the Doshi strategies are the band strategies of type one in which $y_2=y_3$.

Third step. We consider the *band strategies of type two* where the non-action zone has two connected components. Here, the switching zones are of the form $A_{12}=[y_1,y_4]$ and $A_{21}=[l,y_2]$ and the selection zones are of the form $C_1=[l,y_3]$ and $C_2=(y_3,b)$ for $l\leq y_2\leq y_3<y_1<y_4<b$; in these band strategies, the non-action zone $(y_2,y_1)\cup(y_4,b)$ has two connected components. Now, we minimize $W_0^{\mathcal{A}}(b)$ among the four variables $y_2,\ y_3,\ y_1,\ y_4$. Again, we check whether the associated cost functions $W_0^{\mathcal{A}}(b),W_1^{\mathcal{A}}$ and $W_2^{\mathcal{A}}$ of this strategy satisfy the conditions of Theorem 4.4, if they do this is the optimal strategy; if this is not the case, we consider band strategies where the non-action zone has more connected components. And so on...

In the next section, we describe how to find the cost functions of band strategies with one and two connected components using scale functions. We also show how to find the decomposition into the different types of costs: holding, production, switching and penalty costs.

In Section 7, we show examples where the optimal strategy is a Doshi strategy (Figure 1), is a band strategy of type one (Figure 6) and is a band strategy of type two (Figure 11).

6. The value functions of band strategies

In this section we derive the cost functions for band strategies of type one and two. Throughout this section we assume that $l=0$. We further assume that the holding cost per time unit in phase i when the inventory level is x is : $h_i(x)=a_i+c_ix$ for $i=1,2$, where $a_i,c_i\geq 0$ are given. To obtain the value function we apply the fluctuation theory for Lévy processes as described in Chapter 8 in Kyprianou (2014) and Avram et al. (2019).

6.1. Preliminaries

For $i=1,2$ let

$$X_{i,t}:=x+\sigma_i t-\sum_{n=1}^{N(t)}Y_n$$

be the uncontrolled process at phase i with initial inventory level x. The processes X_i are spectrally negative bounded variation Lévy processes. Let us define

$$\varphi_i(\theta) = \log \mathbb{E}\left[e^{\theta(X_{i,1}-x)}\right] = \sigma_i\theta - \lambda + \lambda\mathcal{L}_Y(\theta),$$

where $\mathcal{L}_Y(\theta) := \mathbb{E}[e^{-\theta Y_1}]$. Let us also define the exit times $\tau_{i,a}^- = \inf\{t : X_{i,t} < a\}$ and $\tau_{i,d}^+ = \inf\{t : X_{i,t} = d\}$.

In this section, we use the following notations:

- $W_i^{(q)}(x)$ (the scale function associated with X_i). This scale function is defined by its Laplace transform:

$$\int_0^\infty e^{-\theta x} W_i^{(q)}(x)dx = \frac{1}{q - \varphi_i(\theta)}.$$

- $$Z_i^{(q)}(x,\theta) = e^{\theta x}\left(1 + (q - \varphi_i(\theta))\int_0^x e^{-\theta y} W_i^{(q)}(y)dy\right).$$

  Denote $Z_i^{(q)}(x) = Z_i^{(q)}(x,0) = 1 + q\int_0^x W_i^{(q)}(y)dy$.

- $\overline{W}_i^{(q)}(x) = \int_0^x W_i^{(q)}(y)dy$.
- $\overline{\overline{W}}_i^{(q)}(x) = \int_0^x \overline{W}_i^{(q)}(y)dy$.
- $\overline{Z}_i^{(q)}(x) = \int_0^x Z_i^{(q)}(y)dy = x + q\overline{\overline{W}}_i(x)$.

Throughout this section, we will also use the following results:

- $$\mathbb{E}_x\left[e^{-q\tau_{i,d}^+} 1_{\tau_{i,d}^+ < \tau_{i,a}^-}\right] = \frac{W_i^{(q)}(x-a)}{W_i^{(q)}(d-a)}. \tag{6.1}$$

- $$\mathbb{E}_x\left[e^{-q\tau_{i,0}^- + \theta X_{i,\tau_{i,0}^-}} 1_{\tau_{i,0}^- < \tau_{i,d}^+}\right] = Z_i^{(q)}(x,\theta) - \frac{W_i^{(q)}(x)}{W_i^{(q)}(d)} Z_i^{(q)}(d,\theta). \tag{6.2}$$

- $$\mathbb{E}_x\left[e^{-q\tau_{i,0}^-} 1_{\tau_{i,0}^- < \tau_{i,d}^+}\right] = Z_i^{(q)}(x) - \frac{W_i^{(q)}(x)}{W_i^{(q)}(d)} Z_i^{(q)}(d). \tag{6.3}$$

- For $0 < y < d$, let us define the q-potential measure of X_i as

$$U_i^{(q)}(a,d,x,dy) = \int_0^\infty e^{-qt}\mathbb{P}_x\left[X_{i,t} \in dy, \tau_{i,a}^- \wedge \tau_{i,d}^+ > t\right]dt. \tag{6.4}$$

By Theorem 8.7 in Kyprianou (2014), $U_i^{(q)}(a,d,x,dy) = u_i^{(q)}(x,y)dy$, where

$$u_i^{(q)}(a,d,x,y) = \frac{W_i^{(q)}(x-a)W_i^{(q)}(d-y)}{W_i^{(q)}(d-a)} - W_i^{(q)}(x-y). \tag{6.5}$$

Throughout, we denote by $\mathbb{E}^i$ the expectation according to the probability law $\mathbb{P}^i$ induced by the process X_i for $i = 1, 2$.

6.2. Cost functions for strategies of type one.

As defined in the previous sections the switching zone are $A_{12} = [y_1, b)$ and $A_{21} = [0, y_2]$ and the selection zones are $C_1 = [0, y_3]$ and $C_2 = (y_3, b)$ for $0 \leq y_2 \leq y_3 < y_1 < b$; the non-action zone is (y_2, y_1). The value function is obtained in three steps, first we obtain the expected discounted holding cost, then the expected discounted shortage cost and finally the expected discounted switching cost.

6.2.1. Expected discounted holding cost.

Here, we compute the formulas for:

- $\mathcal{H}_i^{(q)}(x)$– the expected discounted holding cost starting at x at phase i, $i = 1, 2$.
- $\mathcal{H}_0^{(q)}(b)$– the expected discounted holding cost starting at b.
- $H_1^{(q)}(x, y_1)$– the expected discounted holding cost until reaching y_1 starting at x at phase 1, $0 \leq x < y_1$.
- $H_2^{(q)}(x, y_2, b)$– the expected discounted holding cost until reaching b or down-crossing y_2 starting at x at phase 2, $y_2 < x < b$.

In order to do that, let us define $\underline{X}_{1,t} = \inf\{s \leq t, X_{1,s}\}$ and $L_t = -(\underline{X}_{1,t} \wedge 0)$. Let $R_t = X_{1,t} + L_t$. Let $\kappa_{y_1}^+ = \inf\{t : R_t \geq y_1\}$ be the first time that R reaches y_1. Notice that when the inventory is less than y_1 and the phase is 1, the inventory evolves as R. By Theorem 8.1 (ii) in Kyprianou (2014),

$$\mathbb{E}_x^1[e^{-q\kappa_{y_1}^+}] = \frac{Z_1^{(q)}(x)}{Z_1^{(q)}(y_1)}. \tag{6.6}$$

Remark 6.1. *The main tool to evaluate the expected discounted holding cost is the Kella–Whitt martingale, [16]: let X_t be a spectrally negative Lévy process with Laplace exponent $\varphi(\alpha) = \log \mathbb{E}[e^{\alpha(X_1 - X_0)}]$, Y_t an adapted process with bounded expected variation on finite intervals and $V_t = X_t + Y_t$. Let $\Delta Y_s = Y_s - Y_{s^-}$ and Y^c the continuous part of Y, i.e. $Y_t^c = Y_t - \sum_{0 \leq s \leq t} \Delta Y_s$. Then:*

$$\begin{aligned} M_t = {}& \varphi(\alpha) \int_0^t e^{\alpha V_s} ds + e^{\alpha V_0} - e^{\alpha V_t} + \alpha \int_0^t e^{\alpha V_s} dY_s^c \\ &+ \sum_{0 \leq s \leq t} e^{\alpha V_s}(1 - e^{-\alpha \Delta Y_s}) \end{aligned} \tag{6.7}$$

is a zero mean martingale.

From the strong Markov property at y_1 and (6.6), it follows that for $0 < x < y_1$,

$$\mathcal{H}_1^{(q)}(x) = H_1^{(q)}(x, y_1) + \mathbb{E}_x^1[e^{-q\kappa_{y_1}^+}]\mathcal{H}_2^{(q)}(y_1) = H_1^{(q)}(x, y_1) + \frac{Z_1^{(q)}(x)}{Z_1^{(q)}(y_1)}\mathcal{H}_2^{(q)}(y_1). \tag{6.8}$$

Similarly, for $y_2 < x < b$,

$$\begin{aligned} \mathcal{H}_2^{(q)}(x) = {}& H_2^{(q)}(x, y_2, b) + \mathbb{E}_x^2\left[e^{-q\tau_{2,y_2}^-} 1_{\tau_{2,y_2}^- < \tau_{2,b}^+} \mathcal{H}_1^{(q)}(X_{2,\tau_{2,y_2}^-})\right] \\ &+ \mathbb{E}_x\left[e^{-q\tau_b^+} 1_{\tau_{2,b}^+ < \tau_{2,y_2}^-}\right] \mathcal{H}_0^{(q)}(b). \end{aligned} \tag{6.9}$$

And, if we denote by $Exp(\lambda)$ an exponentially distributed random variable with rate λ,

$$\begin{aligned} \mathcal{H}_0^{(q)}(b) = {}& h_0(b)\mathbb{E}\left[\int_0^{Exp(\lambda)} e^{-qt} dt\right] \\ &+ \mathbb{E}\left[e^{-qExp(\lambda)}\left(\int_0^{b-y_3} \mathcal{H}_2^{(q)}(b-z)dF(z) + \int_{b-y_3}^{\infty} \mathcal{H}_1^{(q)}(b-z)dF(z)\right)\right] \\ = {}& \frac{h_0(b)}{q+\lambda} + \frac{\lambda}{\lambda+q}\left(\int_0^{b-y_3} \mathcal{H}_2^{(q)}(b-z)dF(z) + \int_{b-y_3}^{\infty} \mathcal{H}_1^{(q)}(b-z)dF(z)\right). \end{aligned} \tag{6.10}$$

Remark 6.2. *For $x<0$, we define $H_1^{(q)}(x,y_1)=H_1^{(q)}(0,y_1)$ and $\mathcal{H}_1^{(q)}(x)=\mathcal{H}_1^{(q)}(0)$.*

First, we find the formula for $H_2^{(q)}(x,y_2,b)$:

$$H_2^{(q)}(x,y_2,b)=\mathbb{E}_x^2\left[\int_0^{\tau_{2,b}^+\wedge\tau_{2,y_2}^-}e^{-qt}(a_2+c_2X_{2,t})dt\right]$$

$$=\frac{a_2}{q}(1-\mathbb{E}_x^2[e^{-q(\tau_{2,b}^+\wedge\tau_{2,y_2}^-)}])+c_2\mathbb{E}_x^2[\int_0^{\tau_{2,b}^+\wedge\tau_{2,y_2}^-}e^{-qt}X_{2,t}dt]. \tag{6.11}$$

Let

$$h_{2,1}(x,y_2,b)=\frac{1}{q}\left(1-\mathbb{E}_x^2[e^{-q(\tau_{2,b}^+\wedge\tau_{2,y_2}^-)}]\right). \tag{6.12}$$

Applying (6.3) and (6.1) yields:

$$h_{2,1}(x,y_2,b)=\frac{1}{q}\left(1-\mathbb{E}_x^2\left[e^{-q\tau_{2,y_2}^-}1_{\tau_{2,y_2}^-<\tau_{2,b}^+}\right]-\mathbb{E}_x^2\left[e^{-q\tau_{2,b}^+}1_{\tau_{2,b}^+<\tau_{2,y_2}^-}\right]\right)$$

$$=\frac{1}{q}\left(1-Z_2^{(q)}(x-y_2)+\frac{W_2^{(q)}(x-y_2)}{W_2^{(q)}(b-y_2)}Z_2^{(q)}(b-y_2)-\frac{W_2^{(q)}(x-y_2)}{W_2^{(q)}(b-y_2)}\right). \tag{6.13}$$

In order to obtain $\mathbb{E}_x^2[\int_0^{\tau_{2,b}^+\wedge\tau_{2,y_2}^-}e^{-qt}X_{2,t}dt]$, we apply Kella-Whitt martingale (6.7) for $X_{2,t}$, $\varphi_2(\alpha)=\log\mathbb{E}[e^{\alpha(X_{2,1}-X_{2,0})}]$ and $Y_t=-qt/\alpha$, so

$$\mathbb{E}_x^2\left[(\varphi_2(\alpha)-q)\int_0^{\tau_{2,b}^+\wedge\tau_{2,y_2}^-}e^{\alpha X_{2,s}-qs}ds+e^{\alpha x}-e^{\alpha X_{2,\tau_{2,b}^+\wedge\tau_{2,y_2}^-}}\right]=0. \tag{6.14}$$

Taking derivative of (6.14) with respect to α at $\alpha=0$, we obtain

(6.15)

$$\varphi_2'(0)\mathbb{E}_x^2\left[\int_0^{\tau_{2,y_2}^-\wedge\tau_{2,b}^+}e^{-qs}ds\right]+x-\frac{\partial}{\partial\alpha}\mathbb{E}_x^2\left[e^{\alpha X_{2,(\tau_{2,y_2}^-\wedge\tau_{2,b}^+)}-q(\tau_{2,y_2}^-\wedge\tau_{2,b}^+)}|_{\alpha=0}\right]=q\mathbb{E}_x^2\left[\int_0^{\tau_{2,y_2}^-\wedge\tau_{2,b}^+}X_{2,s}e^{-qs}ds\right].$$

By (6.2), (6.14) and (6.15), we get

$$\mathbb{E}_x^2\left[e^{\alpha X_{2,(\tau_{2,y_2}^-\wedge\tau_{2,b}^+)}-q(\tau_{2,y_2}^-\wedge\tau_{2,b}^+)}\right]$$

$$=\mathbb{E}_x^2\left[e^{\alpha b-q\tau_{2,b}^+}1_{\tau_{2,b}^+<\tau_{2,y_2}^-}\right]+\mathbb{E}_x^2\left[e^{\alpha X_{2,\tau_{2,y_2}^-}-q\tau_{2,y_2}^-}1_{\tau_{2,y_2}^-<\tau_{2,b}^+}\right]$$

$$=e^{\alpha b}\frac{W_2^{(q)}(x-y_2)}{W_2^{(q)}(b-y_2)}+e^{\alpha y_2}\left(Z_2^{(q)}(x-y_2,\alpha)-\frac{W_2^{(q)}(x-y_2)}{W_2^{(q)}(b-y_2)}Z_2^{(q)}(b-y_2,\alpha)\right). \tag{6.16}$$

Taking derivative of (6.16) with respect to α at $\alpha=0$, as in (53) of Avram et al. (2019),

$$h_{2,2}(x,y_2,b)=\frac{\partial}{\partial\alpha}\mathbb{E}_x^2[e^{\alpha X_{2,(\tau_{2,y_2}^-\wedge\tau_{2,b}^+)}-q(\tau_{2,y_2}^-\wedge\tau_{2,b}^+)}]|_{\alpha=0}=b\frac{W_2^{(q)}(x-y_2)}{W_2^{(q)}(b-y_2)}$$

$$+y_2\left(Z_2^{(q)}(x-y_2)-\frac{W_2^{(q)}(x-y_2)}{W_2^{(q)}(b-y_2)}Z_2^{(q)}(b-y_2)\right)$$

$$+\overline{Z}_2^{(q)}(x-y_2)-\varphi_2'(0)\overline{W}_2^{(q)}(x-y_2)$$

$$-\frac{W_2^{(q)}(x-y_2)}{W_2^{(q)}(b-y_2)}\left(\overline{Z}_2^{(q)}(b-y_2)-\varphi_2'(0)\overline{W}_2^{(q)}(b-y_2)\right). \tag{6.17}$$

Combining (6.11), (6.13) and (6.17), we have

$$H_2^{(q)}(x,y_2,b)=\left(a_2+\frac{c_2\varphi_2'(0)}{q}\right)h_{2,1}(x,y_2,b)+\frac{c_2}{q}\left(x-h_{2,2}(x,y_2,b)\right). \tag{6.18}$$

Next, we obtain $H_1^{(q)}(x, y_1)$ for $0 \le x < y_1$ –the expected discounted holding cost starting at inventory level x at phase 1 until reaching y_1:

$$H_1^{(q)}(x, y_1) = a_1 \mathbb{E}_x^1 \left[\int_0^{\kappa_{y_1}^+} e^{-qs} ds \right] + c_1 \mathbb{E}_x^1 \left[\int_0^{\kappa_{y_1}^+} e^{-qs} R_s ds \right]. \tag{6.19}$$

By (6.6),

$$\mathbb{E}_x^1 \left[\int_0^{\kappa_{y_1}^+} e^{-qs} ds \right] = \frac{1}{q} \left(1 - \frac{Z_1^{(q)}(x)}{Z_1^{(q)}(y_1)} \right). \tag{6.20}$$

In order to obtain the second term on the right-hand side of (6.19), we apply the Kella-Whitt martingale (6.7) for the process X_1 with $\varphi_1(\alpha) = \log \mathbb{E}[e^{\alpha(X_{1,1} - X_{1,0})}]$, $Y_s = L_s - qs/\alpha$ and $V_s = X_{1,s} + L_s - (q/\alpha)s = R_s - (q/\alpha)s$. Then

$$\begin{aligned} &\mathbb{E}_x^1 \Bigg[(\varphi_1(\alpha) - q) \int_0^{\kappa_{y_1}^+} e^{\alpha R_s - qs} ds + e^{\alpha R_0} - e^{\alpha R_{\kappa_{y_1}^+} - q\kappa_{y_1}^+} \\ &+ \alpha \int_0^{\kappa_{y_1}^+} e^{\alpha R_s - qs} dL_s^c + \sum_{0 \le s \le \kappa_{y_1}^+} e^{\alpha R_s - qs}(1 - e^{-\alpha \Delta L_s}) \Bigg] = 0. \end{aligned} \tag{6.21}$$

Note that $R(\kappa_{y_1}^+) = y_1$ and that $dL_s^c \neq 0$ or $\Delta L_s \neq 0$ implies that $R_s = 0$. Thus (6.21) reduces to

$$\begin{aligned} &\mathbb{E}_x^1 \Bigg[(\varphi_1(\alpha) - q) \int_0^{\kappa_{y_1}^+} e^{\alpha R_s - qs} ds + e^{\alpha x} - e^{\alpha y_1 - q\kappa_{y_1}^+} \\ &+ \alpha \int_0^{\kappa_{y_1}^+} e^{-qs} dL_s^c + \sum_{0 \le s \le \kappa_{y_1}^+} e^{-qs}(1 - e^{-\alpha \Delta L_s}) \Bigg] = 0. \end{aligned} \tag{6.22}$$

By Eq. (80)-(81) in [1],

$$\begin{aligned} &\mathbb{E}_x^1 \left[\int_0^{\kappa_{y_1}^+} e^{-qs} dL_s \right] \\ &= \frac{Z_1^{(q)}(x)}{Z_1^{(q)}(y_1)} \left(\overline{Z}_1^{(q)}(y_1) + \varphi_1'(0))/q \right) - \left(\overline{Z}_1^{(q)}(x) + \varphi_1'(0))/q \right). \end{aligned} \tag{6.23}$$

Taking derivative of (6.22) with respect to α at $\alpha = 0$, and applying (6.6) yields:

$$\begin{aligned} &\varphi_1'(0) \int_0^{\kappa_{y_1}^+} e^{-qs} ds + x - y_1 \frac{Z_1^{(q)}(x)}{Z_1^{(q)}(y_1)} \\ &\frac{Z_1^{(q)}(x)}{Z_1^{(q)}(y_1)} \left(\overline{Z}_1^{(q)}(y_1) + \varphi_1'(0)/q \right) - \left(\overline{Z}_1^{(q)}(x) + \varphi_1'(0)/q \right) \\ &= q \mathbb{E}_x \left[\int_0^{\kappa_{y_1}^+} R_s e^{-qs} ds \right]. \end{aligned} \tag{6.24}$$

Applying (6.6) and after some algebra (6.24) yields:

$$\mathbb{E}_x \left[\int_0^{\kappa_{y_1}^+} R_s e^{-qs} ds \right] = \frac{Z_1^{(q)}(x)}{Z_1^{(q)}(y_1)} \overline{\overline{W}}_1^{(q)}(y_1) - \overline{\overline{W}}_1^{(q)}(x). \tag{6.25}$$

By (6.19), the expected discounted "fixed" part of the holding cost until reaching y_1 is given by

$$a_1 \int_0^{\kappa_{y_1}^+} e^{-qt} dt = \frac{a_1}{q} \left(1 - \frac{Z_1^{(q)}(x)}{Z_1^{(q)}(y_1)} \right).$$

Applying (6.19), (6.20) and (6.25) , we get

$$H_1^{(q)}(x,y_1) = \frac{a_1}{q}\left(1 - \frac{Z_1^{(q)}(x)}{Z_1^{(q)}(y_1)}\right) + c_1\left(\frac{Z_1^{(q)}(x)}{Z_1^{(q)}(y_1)}\overline{\overline{W}}_1^{(q)}(y_1) - \overline{\overline{W}}_1^{(q)}(x)\right). \tag{6.26}$$

In order to obtain $\mathcal{H}_2^{(q)}(x)$ –the expected discounted holding cost starting at inventory level x at phase 2-, we first derive $\mathbb{E}_x^2\left[e^{-q\tau_{2,y_2}^-}1_{\tau_{2,y_2}^-<\tau_b^+}\mathcal{H}_1^{(q)}(X_{2,\tau_{2,y_2}^-})\right]$.

For a function g satisfying the conditions of Theorem 2 in Loeffen (2018), let us define

$$\Omega^2(g(x)) = \mathbb{E}_x^2[e^{-q\tau_{2,y_2}^-}g(X_{2,\tau_{2,y_2}^-})1_{\tau_{2,y_2}^-<\tau_{2,b}^+}].$$

Then, by the aforementioned Theorem 2,

$$\Omega^2(g(x)) = g(x) - \frac{W_2^{(q)}(x-y_2)}{W_2^{(q)}(b-y_2)}g(b) + \int_{y_2}^b(\mathcal{G}_2-q)g(z)\left(\frac{W_2^{(q)}(x-y_2)}{W_2^{(q)}(b-y_2)}W_2^{(q)}(b-z) - W_2^{(q)}(x-z)\right)dz, \tag{6.27}$$

where $\mathcal{G}_2$ is the infinitesimal generator of X_2. Let us first find $\Omega^2(Z_1^q)(x)$: if $\mathcal{G}_1$ is the infinitesimal generator of X_1, then

$$(\mathcal{G}_2-\mathcal{G}_1)g(x) = (\sigma_2-\sigma_1)g'(x). \tag{6.28}$$

It is well known that

$$(\mathcal{G}_1-q)Z_1^{(q)}(x) = 0. \tag{6.29}$$

Thus equations (6.27) and (6.28) yield:

$$\begin{aligned}
\Omega^2(Z_1^{(q)}(x)) &= Z_1^{(q)}(x) - \frac{W_2^{(q)}(x-y_2)}{W_2^{(q)}(b-y_2)}Z_1^{(q)}(b) \\
&+ \int_{y_2}^b\Big((\mathcal{G}_1-q)Z_1^{(q)}(z) + (\sigma_2-\sigma_1)qW_1^{(q)}(x)\Big)\left[\frac{W_2^{(q)}(x-y_2)}{W_2^{(q)}(b-y_2)}W_2^{(q)}(b-z) - W_2^{(q)}(x-z)\right]dz \\
&= Z_1^{(q)}(x) - \frac{W_2^{(q)}(x-y_2)}{W_2^{(q)}(b-y_2)}Z_1^{(q)}(b) \\
&+ \int_{y_2}^b(\sigma_2-\sigma_1)qW_1^{(q)}(z)\left[\frac{W_2^{(q)}(x-y_2)}{W_2^{(q)}(b-y_2)}W_2^{(q)}(b-z) - W_2^{(q)}(x-z)\right]dz.
\end{aligned} \tag{6.30}$$

Similarly, let us consider

$$\Omega^2(\overline{\overline{W}}_1^{(q)}(x)) = \mathbb{E}_x^2\left[e^{-q\tau_{2,y_2}^-}1_{\tau_{2,y_2}^-<\tau_{2,b}^+}\overline{\overline{W}}_1^{(q)}(X(\tau_{2,y_2}^-))\right], \tag{6.31}$$

by (6.29) we have

$$(\mathcal{G}_1-q)\overline{\overline{W}}_1^{(q)}(x) = x. \tag{6.32}$$

Thus,

$$\Omega^2(\overline{\overline{W}}_1^{(q)}(x)) = \overline{\overline{W}}_1^{(q)}(x) - \frac{W_2^{(q)}(x-y_2)}{W_2^{(q)}(b-y_2)}\overline{\overline{W}}_1^{(q)}(b)$$
$$+ \int_{y_2}^{b} (\mathcal{G}_1 - q)\overline{\overline{W}}_1^{(q)}(z) + (\sigma_2 - \sigma_1)\overline{W}_1^{(q)}(z)) \left[\frac{W_2^{(q)}(x-y_2)}{W_2^{(q)}(b-y_2)} W_2^{(q)}(b-z) - W_2^{(q)}(x-z)\right] dz$$
$$= \overline{\overline{W}}_1^{(q)}(x) - \frac{W_2^{(q)}(x-y_2)}{W_2^{(q)}(b-y_2)}\overline{\overline{W}}_1^{(q)}(b)$$
$$+ \int_{y_2}^{b} \left(z + (\sigma_2 - \sigma_1)\overline{W}_1^{(q)}(z)\right) \left[\frac{W_2^{(q)}(x-y_2)}{W_2^{(q)}(b-y_2)} W_2^{(q)}(b-z) - W_2^{(q)}(x-z)\right] dz.$$

Equations (6.8), (6.26), (6.9) and (6.3) yield:

$$\begin{aligned} \mathcal{H}_2^{(q)}(x) &= H_2(x, y_2, b) \\ &+ \frac{a_1}{q}\left(Z_2^{(q)}(x-y_2) - \frac{W_2^{(q)}(x-y_2)}{W_2^{(q)}(b-y_2)} Z_2^{(q)}(b-y_2) - \frac{\Omega^2(Z_1^{(q)}(x))}{Z_1^{(q)}(y_1)}\right) \\ &+ c_1\left(\frac{\Omega^2(Z_1^{(q)}(x))}{Z_1^{(q)}(y_1)}\overline{\overline{W}}_1^{(q)}(y_1) - \Omega^2(\overline{\overline{W}}_1^{(q)}(x))\right) \\ &+ \frac{\Omega^2(Z_1^{(q)}(x))}{Z_1^{(q)}(y_1)}\mathcal{H}_2^{(q)}(y_1) + \frac{W_2^{(q)}(x-y_2)}{W_2^{(q)}(b-y_2)}\mathcal{H}_0^{(q)}(b). \end{aligned} \tag{6.33}$$

Let

$$\begin{aligned} A(x) &:= H_2^{(q)}(x, y_2, b) \\ &+ \frac{a_1}{q}\left(Z_2^{(q)}(x-y_2) - \frac{W_2^{(q)}(x-y_2)}{W_2^{(q)}(b-y_2)} Z_2^{(q)}(b-y_2) - \frac{\Omega^2(Z_1^{(q)}(x))}{Z_1^{(q)}(y_1)}\right) \\ &+ c_1\left(\frac{\Omega^2(Z_1^{(q)}(x))}{Z_1^{(q)}(y_1)}\overline{\overline{W}}_1^{(q)}(y_1) - \Omega^2(\overline{\overline{W}}_1^{(q)}(x))\right), \end{aligned} \tag{6.34}$$

then

$$\mathcal{H}_2^{(q)}(x) = A(x) + \frac{\Omega^2(Z_1^{(q)}(x))}{Z_1^{(q)}(y_1)}\mathcal{H}_2^{(q)}(y_1) + \frac{W_2^{(q)}(x-y_2)}{W_2^{(q)}(b-y_2)}\mathcal{H}_0^{(q)}(b). \tag{6.35}$$

Substituting $x = y_1$ in (6.35) and solving for $\mathcal{H}_2^{(q)}(y_1)$ yield:

$$\mathcal{H}_2^{(q)}(y_1) = \frac{A(y_1) + \frac{W_2^{(q)}(y_1-y_2)}{W_2^{(q)}(b-y_2)}\mathcal{H}_0^{(q)}(b)}{1 - \frac{\Omega^2(Z_1^{(q)}(y_1))}{Z_1^{(q)}(y_1)}}. \tag{6.36}$$

Equations (6.33)-(6.36) yield that

$$\mathcal{H}_2^{(q)}(x) = \alpha_2(x)\mathcal{H}_0^{(q)}(b) + \beta_2(x), \tag{6.37}$$

where

$$\alpha_2(x) = \frac{W_2^{(q)}(x-y_2)}{W_2^{(q)}(b-y_2)} + \frac{\Omega^2(Z_1^{(q)}(x))}{Z_1^{(q)}(y_1) - \Omega^2(Z_1^{(q)}(y_1))}\frac{W_2^{(q)}(y_1-y_2)}{W_2^{(q)}(b-y_2)} \tag{6.38}$$

$$\beta_2(x) = A(x) + \frac{\Omega^2(Z_1^{(q)}(y_1))A(y_1)}{Z_1^{(q)}(y_1) - \Omega^2(Z_1^{(q)}(y_1))}. \tag{6.39}$$

Substituting (6.37) in (6.8), we get

$$\begin{aligned}\mathcal{H}_1^{(q)}(x) &= H_1^{(q)}(x,y_1) + \frac{Z_1^{(q)}(x)}{Z_1^{(q)}(y_1)}\alpha_2(y_1)\mathcal{H}_0^{(q)}(b) + \beta_2(y_1)\frac{Z_1^{(q)}(x)}{Z_1^{(q)}(y_1)} \\ &= \alpha_1(x)\mathcal{H}_0^{(q)}(b) + \beta_1(x), \end{aligned} \tag{6.40}$$

where

$$\alpha_1(x) = \frac{Z_1^{(q)}(x)}{Z_1^{(q)}(y_1)}\alpha_2(y_1)$$

$$\beta_1(x) = H_1^{(q)}(x,y_1) + \frac{Z_1^{(q)}(x)}{Z_1^{(q)}(y_1)}\beta_2(y_1). \tag{6.41}$$

In order to obtain $\mathcal{H}_i^{(q)}$ for $i=1,2$, we substitute (6.41) and (6.37) in (6.10) and get the following linear equation for $\mathcal{H}_0^{(q)}(b)$.

$$\begin{aligned}\mathcal{H}_0^{(q)}(b) &= \frac{h_0(b)}{q+\lambda} + \frac{\lambda}{\lambda+q}\left(\int_0^{b-y_3}\mathcal{H}_2^{(q)}(b-z)dF(z) + \int_{b-y_3}^{\infty}\mathcal{H}_1^{(q)}(b-z)dF(z)\right) \\ &= \frac{h_0(b)}{q+\lambda} + \frac{\lambda}{\lambda+q}\left(\int_0^{b-y_3}\beta_2(b-z)dF(z) + \int_{b-y_3}^{\infty}\beta_1(b-z)dF(z)\right) \\ &+ \mathcal{H}_0^{(q)}(b)\left(\int_0^{b-y_3}\alpha_2(b-z)dF(z) + \int_{b-y_3}^{\infty}\alpha_1(b-z)dF(z)\right). \end{aligned} \tag{6.42}$$

We obtain $\mathcal{H}_0^{(q)}(b)$ solving the linear equation (6.42); from this, we get $\mathcal{H}_i^{(q)}$ for $i=1,2$.

6.2.2. Expected discounted shortage cost.

Here, we derive formulas for the expected discounted shortage cost $\mathcal{S}_i^{(q)}(x)$ starting at inventory level x at phase i for $i=1,2$ together with the expected discounted shortage cost $\mathcal{S}_0^{(q)}(b)$ starting at inventory level b.

Let us define $S_1^{(q)}(x,y_1)$ as the expected discounted shortage cost starting at inventory level $x \in [0,y_1)$ until the inventory level reaches y_1. By (6.6), we can write

$$\mathcal{S}_1^{(q)}(x) = S_1^{(q)}(x,y_1) + \mathbb{E}_x^1[e^{-\kappa_{y_1}^+}]\mathcal{S}_2^{(q)}(y_1) = S_1^{(q)}(x,y_1) + \frac{Z_1^{(q)}(x)}{Z_1^{(q)}(y_1)}\mathcal{S}_2^{(q)}(y_1). \tag{6.43}$$

Equations (6.5) and (6.1) yield

$$\begin{aligned}\mathcal{S}_2^{(q)}(x) &= \int_{y_2}^{b} u_2^{(q)}(y_2,b,x,z)\lambda\left(\int_{v=z}^{\infty}\left(p(v-z) + \mathcal{S}_1^{(q)}(0)\right)dF(v)\right)dz \\ &+ \int_{y_2}^{b} u_2^{(q)}(y_2,b,x,z)\lambda\left(\int_{v=z-y_2}^{z}\mathcal{S}_1^{(q)}(z-v)dF(v)\right)dz \\ &+ \frac{W_2^{(q)}(x-y_2)}{W_2^{(q)}(b-y_2)}\mathcal{S}_0^{(q)}(b). \end{aligned} \tag{6.44}$$

And also,

$$\begin{aligned}\mathcal{S}_0^{(q)}(b) &= \frac{\lambda}{\lambda+q}\left(\int_b^{\infty}(p(z-b) + \mathcal{S}_1^{(q)}(0))dF(z)\right. \\ &\left. + \int_0^{b-y_3}\mathcal{S}_2^{(q)}(b-z)dF(z) + \int_{b-y_3}^{b}\mathcal{S}_1^{(q)}(b-z)dF(z)\right). \end{aligned} \tag{6.45}$$

First we consider $S_1^{(q)}(x, y_1)$ which corresponds to the expected discounted shortage cost starting at x at phase 1, $0 < x < y_1$ until the process reaches y_1. The definition of $u_1^{(q)}$ given in (6.5) yields

$$S_1^{(q)}(x, y_1) = \int_0^{y_1} u_1^{(q)}(0, y_1, x, z)\lambda \left(\int_{v=z}^{\infty} p(v-z)dF(v)\right) dz \tag{6.46}$$

$$+ \mathbb{E}_x^1 \left(e^{-q\tau_{1,0}^-} 1_{\tau_{1,0}^- < \tau_{1,y_1}^+}\right) \cdot \int_0^{y_1} u_1^{(q)}(0, y_1, 0, z)\lambda \left(\int_{v=z}^{\infty} p(v-z)dF(v)\right) dz \tag{6.47}$$

$$\cdot \sum_{j=0}^{\infty} \left(\mathbb{E}_0^1 [e^{-q\tau_{1,0}^-} 1_{\tau_{1,0}^- < \tau_{1,y_1}^+}]\right)^j,$$

where (6.46) describes the expected discounted shortage cost occurring before the inventory level reaches y_1, and (6.47) describes the expected discounted shortage costs occurring after the first downcrossing level 0. Applying equation (6.3) yields:

$$S_1^{(q)}(x, y_1) = \int_0^{y_1} u_1^{(q)}(0, y_1, x, z)\lambda \left(\int_{v=z}^{\infty} p(v-z)dF(v)\right) dz$$

$$+ \frac{Z^{(q)}(x) - \frac{W_1^{(q)}(x)}{W_1^{(q)}(y_1)} Z_1^{(q)}(y_1)}{\frac{W_1^{(q)}(0)}{W_1^{(q)}(y_1)} Z_1^{(q)}(y_1)} \int_0^{y_1} u_1^{(q)}(0, y_1, 0, z)\lambda \left(\int_{v=z}^{\infty} p(v-z)dF(v)\right) dz. \tag{6.48}$$

Next, we obtain linear equations to obtain $\mathcal{S}_i^{(q)}(x)$ and $\mathcal{S}_0^{(q)}(b)$. Let us define

$$\mu(x) := \int_{y_2}^{b} u_2^{(q)}(y_2, b, x, z)\lambda \left(\int_{v=z}^{\infty} p(v-z)dF(z)\right) dz$$

$$+ \lambda S_1^{(q)}(0, y_1) \int_{y_2}^{b} u_2^{(q)}(y_2, b, x, z)\overline{F}(z)dz$$

$$+ \int_{y_2}^{b} u_2^{(q)}(y_2, b, x, z)\lambda \left(\int_{v=z-y_2}^{z} S_1^{(q)}(z-v, y_1)dF(v)\right) dz, \tag{6.49}$$

where $\bar{F}(z) = 1 - F(z)$. Let us also define

$$\gamma(x) := \lambda \frac{1}{Z_1^{(q)}(y_1)} \int_{y_2}^{b} u_2^{(q)}(y_2, b, x, z)\overline{F}(z)dz$$

$$+ \lambda \int_{y_2}^{b} u_2^{(q)}(y_2, b, x, z) \left(\int_{v=z-y_2}^{z} \frac{Z_1^{(q)}(z-v)}{Z_1^{(q)}(y_1)} dF(z)\right) dz. \tag{6.50}$$

Substituting (6.43) in (6.44), we have that $\mathcal{S}_2^{(q)}(x)$ can be written as follows:

$$\mathcal{S}_2^{(q)}(x) = \mu(x) + \gamma(x)\mathcal{S}_2^{(q)}(y_1) + \frac{W_2^{(q)}(x-y_2)}{W_2^{(q)}(b-y_2)} \mathcal{S}_0^{(q)}(b). \tag{6.51}$$

Solving (6.51) for $x = y_1$ yield

$$\mathcal{S}_2^{(q)}(y_1) = \frac{\mu(y_1) + \frac{W_2^{(q)}(y_1-y_2)}{W_2^{(q)}(b-y_2)} \mathcal{S}_0^{(q)}(b)}{1 - \gamma(y_1)}. \tag{6.52}$$

Let us define

$$\mu_2(x) := \mu(x) + \gamma(x) \frac{\mu(y_1)}{1 - \gamma(y_1)},$$

$$\gamma_2(x) := \frac{W_2^{(q)}(x-y_2)}{W_2^{(q)}(b-y_2)} + \frac{W_2^{(q)}(y_1-y_2)}{W_2^{(q)}(b-y_2)} \frac{\gamma(x)}{1-\gamma(y_1)},$$

and

$$\mu_1(x) := S_1^{(q)}(x, y_1) + \frac{Z_1^{(q)}(x)}{Z_1^{(q)}(y_1)}\mu_2(y_1),$$

$$\gamma_1(x) := \frac{Z_1^{(q)}(x)}{Z_1^{(q)}(y_1)}\gamma_2(y_1). \tag{6.53}$$

Thus, equations (6.44), (6.43), (6.51) and (6.52) yield:

$$\mathcal{S}_i^{(q)}(x) = \mu_i(x) + \mathcal{S}_0^{(q)}(b)\gamma_i(x) \text{ for } i = 1, 2. \tag{6.54}$$

Substituting (6.54) in (6.45), we get the following linear equation for $\mathcal{S}_0(b)$,

$$\mathcal{S}_0^{(q)}(b) = \frac{\lambda}{\lambda + q}\left(\int_b^{\infty}(p(z-b) + \mu_1(0) + \gamma_1(0)\mathcal{S}_0^{(q)}(b))dF(z)\right.$$
$$\left. + \int_0^{b-y_3}(\mu_2(b-z) + \mathcal{S}_0^{(q)}(b)\gamma_2(b-z))dF(z) + \int_{b-y_3}^{b}(\mu_1(b-z) + \gamma_1(b-z)\mathcal{S}_0^{(q)}(b))dF(z)\right),$$

and so

$$\mathcal{S}_0^{(q)}(b) = \frac{\frac{\lambda}{\lambda+q}\left(\int_b^{\infty}(p(z-b) + \mu_1(0))dF(z) + \int_0^{b-y_3}\mu_2(b-z)dF(z) + \int_{b-y_3}^{b}\mu_1(b-z)dF(z)\right)}{1 - \gamma_1(0)\overline{F}(b) + \int_0^{b-y_3}\gamma_2(b-z)dF(z) + \int_{b-y_3}^{b}\gamma_1(b-z)dF(z)}. \tag{6.55}$$

Finally, from (6.54), we get the formulas for $\mathcal{S}_i^{(q)}(x)$ for $i = 1, 2$.

6.2.3. Expected discounted switching cost.

Here, we compute the formulas for the expected discounted switching cost. Let $\mathcal{K}_i^{(q)}(x)$, $i = 1, 2$ be the expected discounted switching cost starting at inventory level x and phase i for $i = 1, 2$ and let $\mathcal{K}_0^{(q)}(b)$ be the expected discounted switching cost starting at b. Assume that initially the inventory level x is at phase 1, then the first switching from phase 1 to phase 2 occurs at $\kappa_{y_1}^+$. By (6.6),

$$\mathcal{K}_1^{(q)}(x) = \mathbb{E}_x^1[e^{-a\kappa_{y_1}^+}]\left(K_{12} + \mathcal{K}_2^{(q)}(y_1)\right) = \frac{Z_1^{(q)}(x)}{Z_1^{(q)}(y_1)}\left(K_{12} + \mathcal{K}_2^{(q)}(y_1)\right). \tag{6.56}$$

If initially the inventory level x is at phase 2, then the first switching from phase 2 to phase 1 occurs when the inventory level downcrosses y_2 before reaching b. If the inventory reaches b before downcrossing y_2, there is a switching from phase 2 to phase 0. By (6.1),

$$\mathcal{K}_2^{(q)}(x) = \frac{W_2^{(q)}(x-y_2)}{W_2^{(q)}(b-y_2)}\left(K_{20} + \mathcal{K}_0^{(q)}(b)\right) + \mathbb{E}_x^2\left[e^{-q\tau_{2,y_2}^-}1_{\tau_{2,y_2}^- < \tau_{2,b}^+}(K_{21} + \mathcal{K}_1^{(q)}(X_{2,\tau_{2,y_2}^-}))\right]. \tag{6.57}$$

Due to equations (6.3), (6.56) and (6.30),

$$\mathcal{K}_2^{(q)}(x) = \frac{W_2^{(q)}(x-y_2)}{W_2^{(q)}(b-y_2)}\left(K_{20} + \mathcal{K}_0^{(q)}(b)\right) + \frac{\Omega^2(Z_1^{(q)}(x))}{Z_1^{(q)}(y_1)}\left(K_{12} + \mathcal{K}_2^{(q)}(y_1)\right)$$
$$+ K_{21}\left(Z_2^{(q)}(x-y_2) - \frac{W_2^{(q)}(x-y_2)}{W_2^{(q)}(b-y_2)}Z_2^{(q)}(b-y_2)\right). \tag{6.58}$$

Substituting x by y_1 in (6.58) and solving for $\mathcal{K}_2^{(q)}(y_1)$ yields:

$$\mathcal{K}_2^{(q)}(y_1) = \frac{1}{1 - \frac{\Omega^2(Z_1^{(q)}(y_1))}{Z_1^{(q)}(y_1)}}\left(K_{20}\frac{W_2^{(q)}(y_1-y_2)}{W_2^{(q)}(b-y_2)} + K_{12}\frac{\Omega^2(Z_1^{(q)}(y_1))}{Z_1^{(q)}(y_1)}\right.$$
$$+ K_{21}(Z_2^{(q)}(y_1-y_2) - \frac{W_2^{(q)}(y_1-y_2)}{W_2^{(q)}(b-y_2)}Z_2^{(q)}(b-y_2))$$
$$\left. + \frac{W_2^{(q)}(y_1-y_2)}{W_2^{(q)}(b-y_2)}\mathcal{K}_0^{(q)}(b)\right). \tag{6.59}$$

Thus

$$\mathcal{K}_2^{(q)}(x) = \omega_2(x) + \delta_2(x)\mathcal{K}_0^{(q)}(b), \tag{6.60}$$

where,

$$\begin{aligned}\omega_2(x) &:= K_{20}\frac{W_2^{(q)}(x-y_2)}{W_2^{(q)}(b-y_2)} + K_{21}\left(Z_2^{(q)}(x-y_2) - \frac{W^{(q)}(x-y_2)}{W^{(q)}(b-y_2)}Z_2^{(q)})(b-y_2)\right)\\ &+ \frac{\Omega^2(Z_1^{(q)}(x))}{Z_1^{(q)}(y_1)}\Bigg(K_{12}\\ &+ \frac{K_{20}\frac{W_2^{(q)}(y_1-y_2)}{W_2^{(q)}(b-y_2)} + K_{12}\frac{\Omega^2(Z_1^{(q)}(y_1))}{Z_1^{(q)}(y_1)} + K_{21}\left(Z_2^{(q)}(y_1-y_2) - \frac{W_2^{(q)}(y_1-y_2)}{W_2^{(q)}(b-y_2)}Z_2^{(q)}(b-y_2)\right)}{1-\frac{\Omega^2(Z_1^{(q)}(y_1))}{Z_1^{(q)}(y_1)}}\Bigg)\end{aligned} \tag{6.61}$$

and

$$\delta_2(x) := \frac{W_2^{(q)}(x-y_2)}{W_2^{(q)}(b-y_2)} + \frac{\frac{\Omega^2(Z_1^{(q)}(x))}{Z_1^{(q)}(y_1)}\frac{W_2^{(q)}(y_1-y_2)}{W_2^{(q)}(b-y_2)}}{1-\frac{\Omega^2(Z_1^{(q)}(y_1))}{Z_1^{(q)}(y_1)}}. \tag{6.62}$$

By (6.56),

$$\mathcal{K}_1^{(q)}(x) = \omega_1(x) + \delta_1(x)\mathcal{K}_0^{(q)}(b), \tag{6.63}$$

where

$$\omega_1(x) := \frac{Z_1^{(q)}(x)}{Z_1^{(q)}(y_1)}\left(K_{12} + \omega_2(y_1)\right)$$

and

$$\delta_1(x) := \frac{Z_1^{(q)}(x)}{Z_1^{(q)}(y_1)}\delta_2(y_1). \tag{6.64}$$

Moreover, $\mathcal{K}_0^{(q)}(b)$ satisfies the following linear equation:

$$\begin{aligned}\mathcal{K}_0^{(q)}(b) &= \frac{\lambda}{q+\lambda}\Bigg(\int_0^{b-y_3}(K_{02}+\mathcal{K}_2^{(q)}(b-z))dF(z)\\ &+ \int_{b-y_3}^{b}(K_{01}+\mathcal{K}_1^{(q)}(b-z))dF(z) + \int_b^\infty(K_{01}+\mathcal{K}_1^{(q)}(0))dF(z)\Bigg)\\ &= \frac{\lambda}{q+\lambda}\Bigg(\int_0^{b-y_3}(K_{02}+\omega_2(b-z)+\delta_2(b-z)\mathcal{K}_0^{(q)}(b))dF(z)\\ &+ \int_{b-y_3}^{b}(K_{01}+\omega_1(b-z)+\delta_1(b-z)\mathcal{K}_0^{(q)}(b))dF(z)\\ &+ \int_b^\infty(K_{01}+\omega_1(0)+\delta_1(0)\mathcal{K}_0^{(q)}(b))dF(z)\Bigg),\end{aligned} \tag{6.65}$$

thus

$$\mathcal{K}_0^{(q)}(b) = \frac{\frac{\lambda}{\lambda+q}\left(\int_0^{b-y_3}(K_{02}+\omega_2(b-z))dF(z) + \int_{b-y_3}^{b}(K_{01}+\omega_1(b-z))dF(z) + (K_{01}+\omega_1(0))\overline{F}(b)\right)}{1-\frac{\lambda}{\lambda+q}\left(\int_0^{b-y_3}\delta_2(b-z)dF(z) + \int_{b-y_3}^{b}\delta_1(b-z)dF(z) + \delta_1(0)\overline{F}(b)\right).}. \tag{6.66}$$

6.2.4. Total discounted cost.

As a summary, we have that the total discounted cost starting at inventory level x and phase i is:

Inventory level	Phase	Expected discounted cost
$0 \le x < y_1$	1	$\mathcal{H}_1^{(q)}(x) + \mathcal{S}_1^{(q)}(x) + \mathcal{K}_1^{(q)}(x)$
$0 \le x \le y_2$	2	$\mathcal{H}_1^{(q)}(x) + \mathcal{S}_1^{(q)}(x) + \mathcal{K}_1^{(q)}(x) + K_{21}$
$y_2 < x < b$	2	$\mathcal{H}_2^{(q)}(x) + \mathcal{S}_2^{(q)}(x) + \mathcal{K}_2^{(q)}(x)$
$y_1 \le x < b$	1	$\mathcal{H}_2^{(q)}(x) + \mathcal{S}_2^{(q)}(x) + \mathcal{K}_2^{(q)}(x) + K_{12}$
$x = b$	0	$\mathcal{H}_0^{(q)}(b) + \mathcal{S}_0^{(q)}(b) + \mathcal{K}_0^{(q)}(b)$

6.3. Cost functions for strategies of type two.

Here the switching zone from 1 to 2 is $A_{12} = [y_1, y_4]$, the switching zone from 2 to 1 is $A_{21} = [0, y_2]$, the selection zones are $C_1 = [0, y_3]$ and $C_2 = (y_3, b)$ and the non-action zone $(y_2, y_1) \cup (y_4, b)$ for $0 \le y_2 \le y_3 < y_1 < y_4 < b$.

The analysis of the value function in this case is very similar to the analysis of the strategy of type one. The only difference is in the case when initially the inventory level is $x \in (y_4, b)$ at phase 1. Thus we consider only this case.

Let us start with $\bar{\mathcal{H}}_1^{(q)}(x)$ —the expected discounted holding cost in the case $y_4 < x < b$. Consider the expected discounted holding until reaching b or down-crossing y_4,

$$H_1^{(q)}(x, y_4, b) = \mathbb{E}_x^1 \left[\int_0^{\tau_{1,y_4}^- \wedge \tau_{1,b}^+} e^{-qs}(a_1 + c_1 X_{1,s}) ds \right]. \tag{6.67}$$

Similarly to equations (6.11)-(6.18), we have

$$H_1^{(q)}(x, y_4, b) = \left(a_1 + \frac{c_1 \varphi_1'(0)}{q} \right) h_{1,1}(x, y_4, b) + \frac{c_1}{q} \left(x - h_{1,2}(x, y_4, b) \right), \tag{6.68}$$

where

$$\begin{aligned} h_{1,1}(x, y_4, b) &= \frac{1}{q} \left(1 - \mathbb{E}_x^1 \left[e^{-q\tau_{1,y_4}^-} 1_{\tau_{1,y_4^-} < \tau_{1.b}^+} \right] - \mathbb{E}_x^1 \left[e^{-q\tau_{1,b}^+} 1_{\tau_{1.b}^+ < \tau_{1,y_4^-}} \right] \right) \\ &= \frac{1}{q} \left(1 - Z_1^{(q)}(x - y_4) + \frac{W_1^{(q)}(x - y_4)}{W_1^{(q)}(b - y_4)} Z_1^{(q)}(b - y_4) - \frac{W_1^{(q)}(x - y_4)}{W_1^{(q)}(b - y_4)} \right) \end{aligned} \tag{6.69}$$

and

$$\begin{aligned} h_{1,2}(x, y_4, b) &= \frac{\partial}{\partial \alpha} \mathbb{E}_x \left[e^{\alpha (X_{1,\tau_{1,y_4}^- \wedge \tau_{1,b}^+}) - q(\tau_{1,y_4}^- \wedge \tau_{1,b}^+)} \right] |_{\alpha=0} = b \frac{W_1^{(q)}(x - y_4)}{W_1^{(q)}(b - y_4)} \\ &+ y_4 \left(Z_1^{(q)}(x - y_4) - \frac{W_1^{(q)}(x - y_4)}{W_1^{(q)}(b - y_4)} Z_1^{(q)}(b - y_4) \right) \\ &+ \overline{Z}_1^{(q)}(x - y_4) - \varphi_1'(0) \overline{W}_1^{(q)}(x - y_4) \\ &- \frac{W_1^{(q)}(x - y_4)}{W_1^{(q)}(b - y_4)} \left(\overline{Z}_1^{(q)}(b - y_4) - \varphi_1'(0) \overline{W}_1^{(q)}(b - y_4) \right). \end{aligned} \tag{6.70}$$

Once the inventory level reaches b, the expected discounted holding cost is $\mathcal{H}_0^{(q)}(b)$. In the case that the inventory level down-crosses y_4 before reaching b there are two scenarios: **1.** If X_{1,τ_{1,y_4}^-} lies in $[y_1, y_4]$, then the expected discounted holding cost is $\mathcal{H}_2^{(q)}\left(X_{1,\tau_{1,y_4}^-} \right)$. **2.** If the inventory level immediately after the jump is $X_{1,\tau^-_{1,y_4}}$ lies in $(-\infty, y_1)$, then the expected discounted holding cost is $\mathcal{H}_1^{(q)}(X_{1,\tau^-_{1,y_4}})$. Thus,

$$\begin{aligned}\bar{\mathcal{H}}_1^{(q)}(x) &= H_1^{(q)}(x, y_4, b) + \frac{W_1^{(q)}(x-y_4)}{W_1^{(q)}(b-y_4)} \cdot \mathcal{H}_0^{(q)}(b) \\ &+ \lambda \int_{y_4}^{b} u_1^{(q)}(y_4, b, x, y) \left(\int_{z=y-y_4}^{y-y_1} \mathcal{H}_2^{(q)}(y-z) dF(z) \right) dy \\ &+ \lambda \int_{y_4}^{b} u_1^{(q)}(y_4, b, x, y) \left(\int_{z=y-y_1}^{y} \mathcal{H}_1^{(q)}(y-z) dF(z) \right) dy \\ &+ \mathcal{H}_1^{(q)}(0) \lambda \int_{y_4}^{b} u_1^{(q)}(y_4, b, x, y) \bar{F}(y) dy. \end{aligned} \tag{6.71}$$

Consider now $\bar{\mathcal{S}}_1^{(q)}(x)$ —the expected discounted shortage cost starting at $x \in (y_4, b)$ at phase 1. If the process reaches b before down-crossing y_4, then the expected discounted shortage cost is $\mathcal{S}_0^{(q)}(b)$. If the inventory level down-crosses y_4 before reaching b, then the shortage cost is $\mathcal{S}_2^{(q)}(X_{1,\tau^-_{1,y_4}})$ in the case that $y_1 \le X_{1,\tau^-_{1,y_4}} \le y_4$, is $\mathcal{S}_1^{(q)}(X_{1,\tau^-_{1,y_4}})$ in the case that $0 \le X_{1,\tau^-_{1,y_4}} < y_1$, and is $p(-X_{1,\tau^-_{1,y_4}}) + \mathcal{S}_1^{(q)}(0)$ in the case that $X_{1,\tau^-_{1,y_4}} < 0$. Applying (6.1) and (6.5) yields:

$$\begin{aligned}\bar{\mathcal{S}}_1^{(q)}(x) &= \frac{W_1^{(q)}(x-y_4)}{W_1^{(q)}(b-y_4)} \mathcal{S}_0^{(q)}(b) \\ &+ \lambda \int_{y_4}^{b} u_1^{(q)}(y_4, b, x, y) \int_{z=y-y_4}^{y-y_1} \mathcal{S}_2^{(q)}(y-z) dF(z) dy \\ &+ \lambda \int_{y_4}^{b} u_1^{(q)}(y_4, b, x, y) \int_{z=y-y_1}^{y} \mathcal{S}_1^{(q)}(y-z) dF(z) dy \\ &+ \lambda \int_{y_4}^{b} u_1^{(q)}(y_4, b, x, y) \int_{z=y}^{\infty} p(z-y) dF(z) dy \\ &+ \mathcal{S}_1^{(q)}(0) \lambda \int_{y_4}^{b} u_1^{(q)}(y_4, b, x, y) \bar{F}(y) dy. \end{aligned} \tag{6.72}$$

Similarly we obtain the expected discounted switching cost $\bar{\mathcal{K}}_1^{(q)}(x)$:

$$\begin{aligned}\bar{\mathcal{K}}_1^{(q)}(x) &= \frac{W_1^{(q)}(x-y_4)}{W_1^{(q)}(b-y_4)} \mathcal{K}_0^{(q)}(b) \\ &+ \lambda \int_{y_4}^{b} u_1^{(q)}(x-y_4, y-y_4) \left(\int_{z=y-y_4}^{y-y_1} \mathcal{K}_2^{(q)}(y-z) dF(z) \right) dy \\ &+ K_{12} \lambda \int_{y_4}^{b} u_1^{(q)}(y_4, b, x, y)(F(y-y_4) - F(y-y_1)) dy \\ &+ \lambda \int_{y_4}^{b} u_1^{(q)}(y_4, b, x, y) \left(\int_{z=y-y_1}^{y} \mathcal{K}_1^{(q)}(y-z) dF(z) \right) dy \\ &+ \mathcal{K}_1^{(q)}(0) \lambda \int_{y_4}^{b} u_1^{(q)}(y_4, b, x, y) \bar{F}(y) dy. \end{aligned} \tag{6.73}$$

7. Examples

In this section, we find the optimal strategies for three different situations. In the first one, the optimal strategy is of Doshi type, in the second is of type one and in the third is of type two.

7.1. First Example: Doshi strategy is optimal

In this example we consider two equal manufacturing units, in phase 1 both units are producing together, in phase 2 only one manufacturing unit is working and in phase 0 none of the units are working. We assume that the cost of shutdown each unit is equal to 1, the cost of restarting each unit is equal to 2 and the rate of production of each unit is equal to 3/2. The cost rate of production of each unit is 1/100, and there is also a fix cost rate (independent of the production) equal to 1/1000. Moreover, the holding cost rate is 1/1000 and the storage capacity is $b = 10$. So we have the following parameters $\sigma_1 = 3$, $\sigma_2 = 3/2$, $K_{12} = 1$, $K_{21} = 2$, $K_{20} = 2$, $K_{10} = 4$, $K_{02} = 2$, $K_{01} = 4$, $h_2(x) = (21 + x)/1000$, $h_1(x) = (41 + x)/1000$, $h_0(b) = (1+b)/1000$. We assume that the rate of arrival of the customer demands is $\lambda = 2$, the demands are distributed as $Exp(1.5)$, the discount rate is $q = 0.1$ and the penalty cost when an amount y of a costumer is lost is given by $p(y) = (80 + 40y)/100$ (here we are taking $l = 0$).

We find that the best Doshi strategy is given by the sets $A_{12} = [y_1, 10)$, $A_{21} = [0, y_2]$, $C_1 = [0, y_2]$ and $C_2 = (y_2, 10)$ with $y_2 = 1.526$ and $y_1 = 5.077$. We check that the value functions of this strategy are viscosity solutions of the equations (4.5) and satisfy the conditions of Theorem 4.4; so the best Doshi strategy is the optimal one. We show this optimal strategy en Figure 1.

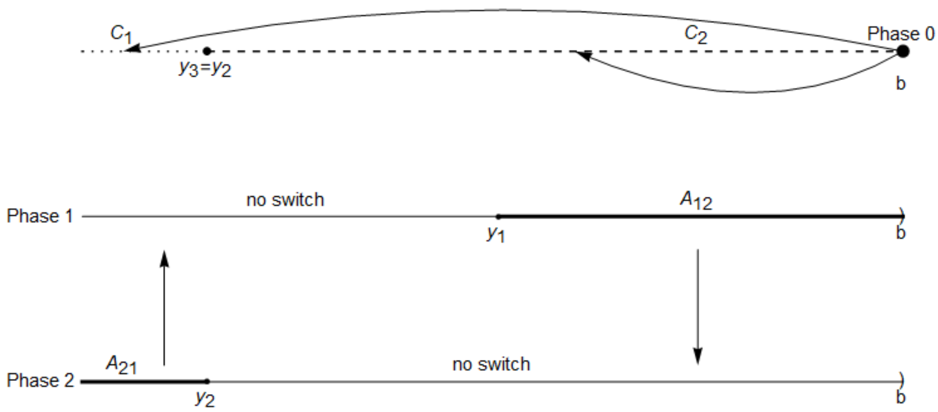

Figure 1: Optimal strategy in first example.

In Figure 2, we show the discounted total cost $V_1(x)$ (dotted), $V_2(x)$ (dashed) and $V_0(b)$ (solid point) of the optimal strategy; in Figure 3, the discounted holding cost $\mathcal{H}_1^{(q)}(x)$ (dotted), $\mathcal{H}_2^{(q)}(x)$ (dashed) and $\mathcal{H}_0^{(q)}(b)$ (solid point) of the optimal strategy; in Figure 4, the discounted penalty cost $\mathcal{S}_1^{(q)}(x)$ (dotted), $\mathcal{S}_2^{(q)}(x)$ (dashed) and $\mathcal{S}_0^{(q)}(b)$ (solid point) of the optimal strategy; and finally in Figure 5, the discounted penalty cost $\mathcal{K}_1^{(q)}(x)$ (dotted), $\mathcal{K}_2^{(q)}(x)$ (dashed) and $\mathcal{K}_0^{(q)}(b)$ (solid point) of the optimal strategy.

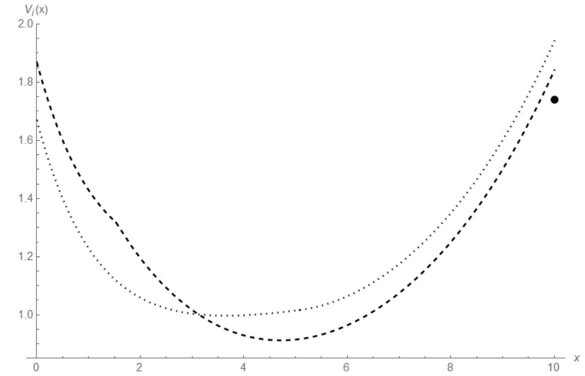

Figure 2: First example, total cost.

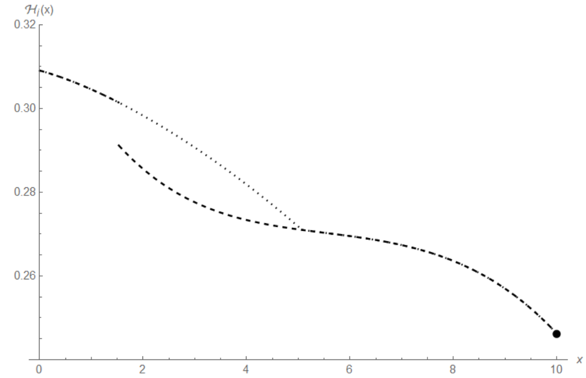

Figure 3: First example, holding cost.

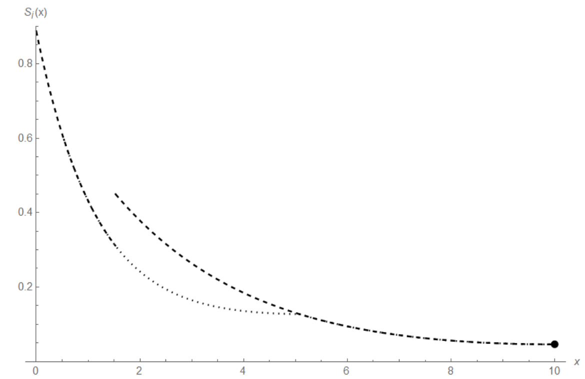

Figure 4: First example, shortage cost.

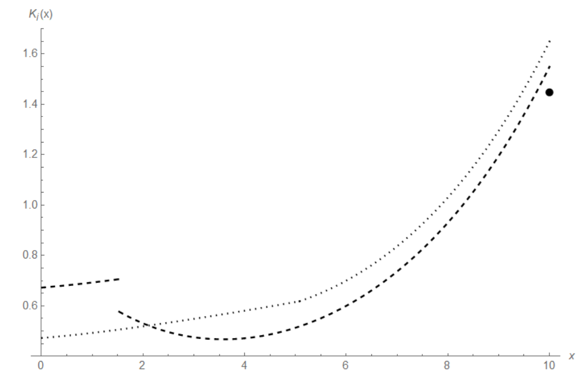

Figure 5: First example, switching cost.

Remark 7.1. *(1) $V_2 - V_1$ is equal to K_{21} in A_{21} and equal to $-K_{12}$ in A_{12}; also $V_2(b^-) - V_0(b) = K_{20}$ and $V_1(b^-) - V_0(b) = K_{12} + K_{10}$. V_2 is not differentiable at y_2, and so it is necessary to use the notion of viscosity solution.*

(2) $\mathcal{H}_2^{(q)} = \mathcal{H}_1^{(q)}$ in the switching zones $A_{21} \cup A_{12}$. $\mathcal{H}_2^{(q)}$ is not continuous at the boundary y_2 between the switching zone A_{21} and the non-action zone (y_2, y_1). The jump of $\mathcal{H}_2^{(q)}$ at y_2 is downward because, for an initial inventory level x in the non-action zone, $X_{1,t} > X_{2,t}$ for $t > 0$ while these processes remain in the non-action zone. Also note that $\mathcal{H}_2^{(q)}(b^-) = \mathcal{H}_1^{(q)}(b^-) = \mathcal{H}_0^{(q)}(b)$.

(3) $\mathcal{S}_2^{(q)} = \mathcal{S}_1^{(q)}$ in the switching zones $A_{21} \cup A_{12}$. As in the previous case and for similar reasons, $\mathcal{S}_2^{(q)}$ has a discontinuity at y_2, but in this case the jump is upward. Also note that $\mathcal{S}_2^{(q)}(b^-) = \mathcal{S}_1^{(q)}(b^-) = \mathcal{S}_0^{(q)}(b)$.

(4) $\mathcal{K}_2^{(q)} - \mathcal{K}_1^{(q)}$ is equal to K_{21} in A_{21} and equal to $-K_{12}$ in A_{12}; also $\mathcal{K}_2^{(q)}(b^-) - \mathcal{K}_0^{(q)}(b) = K_{20}$ and $\mathcal{K}_1^{(q)}(b^-) - \mathcal{K}_0^{(q)}(b) = K_{12} + K_{20}$. $\mathcal{K}_2^{(q)}$ has a downward jump at y_2 because this point is the boundary between the switching zone A_{21} and the non-action zone (y_2, y_1).

7.2. Second Example: Strategy of type one is optimal

In this example, we consider that the demands are distributed as $Exp(1)$, the parameters are $q = 0.1$, $\lambda = 2$, $l = 0$, $b = 20$, the rates of production are $\sigma_1 = 2.5$, $\sigma_2 = 2.2$, and the costs are given by $K_{12} = K_{21} = 0.05$, $K_{20} = 1/200$, $K_{10} = 11/2000$, $K_{01} = K_{02} = 0$, $h_2(x) = (20+x)/1000$, $h_1(x) = (30+x)/1000$, $h_0(b) = (2 + 10b)/10000$, $p(y) = (80 + 40y)/100$.

In this case, the value functions of the best Doshi strategy do not satisfy the condition of Theorem 4.4, so we look for the best band strategies of type one, which is given by the sets $A_{12} = [y_1, 20)$, $A_{21} = [0, y_2]$, $C_1 = [0, y_3]$ and $C_2 = (y_3, 20)$ for $y_2 = 6.213$, $y_3 = 9.805$ and $y_1 = 17.294$. The value functions of this strategy of type one are viscosity solutions of the equations (4.5) and satisfy the conditions of Theorem 4.4, so this is the optimal strategy. We show this optimal strategy in Figure 6.

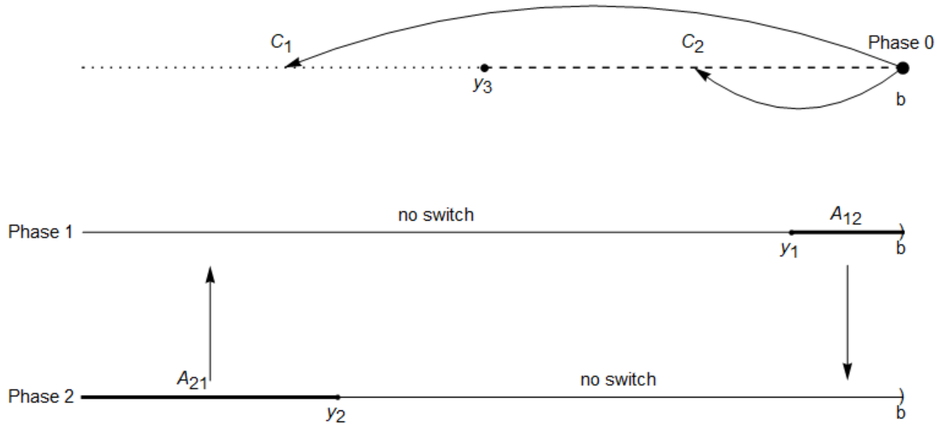

Figure 6: Optimal strategy in second example.

In Figure 7, we show the discounted total cost $V_1(x)$ (dotted), $V_2(x)$ (dashed) and $V_0(b)$ (solid point) of the optimal strategy; in Figure 8, the discounted holding cost $\mathcal{H}_1^{(q)}(x)$ (dotted), $\mathcal{H}_2^{(q)}(x)$ (dashed) and $\mathcal{H}_0^{(q)}(b)$ (solid point) of the optimal strategy; in Figure 9, the discounted penalty cost $\mathcal{S}_1^{(q)}(x)$ (dotted), $\mathcal{S}_2^{(q)}(x)$ (dashed) and $\mathcal{S}_0^{(q)}(b)$ (solid point) of the optimal strategy; and finally in Figure 10, the discounted penalty cost $\mathcal{K}_1^{(q)}(x)$ (dotted), $\mathcal{K}_2^{(q)}(x)$ (dashed) and $\mathcal{K}_0^{(q)}(b)$ (solid point) of the optimal strategy.

The observations of Remark 7.1 hold for this example.

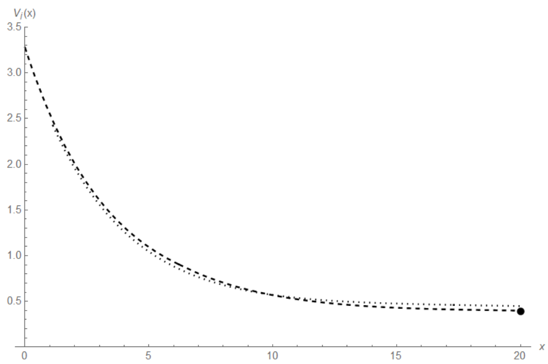

Figure 7: Second example, total cost.

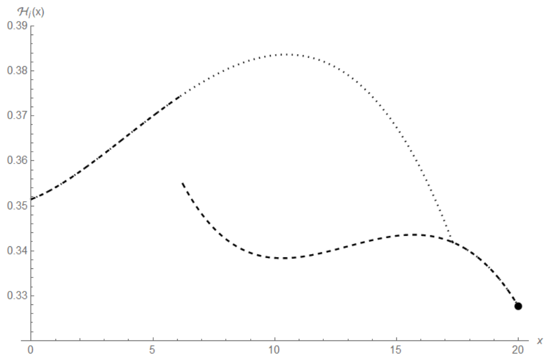

Figure 8: Second example, holding cost.

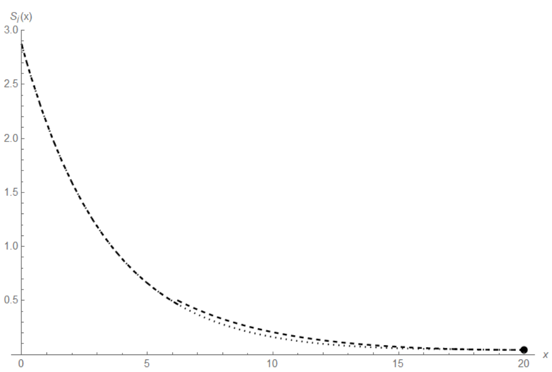

Figure 9: Second example, shortage cost.

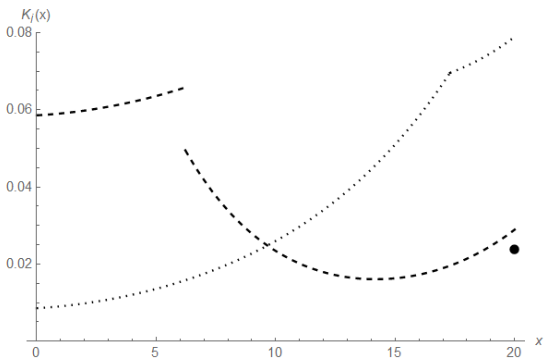

Figure 10: Second example, switching cost.

7.3. Third Example: Strategy of type two is optimal

In this last example, we consider that the demands are distributed as $Exp(1)$, the parameters are $q = 0.1$, $\lambda = 2$, $l = 0$, $b = 10$, the rates of production are $\sigma_1 = 3.5$, $\sigma_2 = 2.5$, and the costs are given by $K_{12} = K_{21} = 0.05$, $K_{20} = 0$, $K_{10} = 1/100$, $K_{01} = K_{02} = 0$, $h_1(x) = h_2(x) = (1 + 12x)/100$, $h_0(b) = (1 + 10b)/100$, $p(y) = 2 + 1.1y$.

In this case, the value functions of the best strategy of type one do not satisfy the condition of Theorem 4.4, so we look for the best band strategies of type two, which is given by the sets $A_{12} = [y_1, y_4]$, $A_{21} = [0, y_2]$, $C_1 = [0, y_3]$ and $C_2 = (y_3, 10)$ for $y_2 = 2.468$, $y_3 = 3.114$, $y_1 = 4.610$, $y_4 = 7.660$. The value functions of this strategy of type two are viscosity solutions of the equations (4.5) and satisfy the conditions of Theorem 4.4, so this is the optimal strategy. We show this optimal strategy in Figure 11.

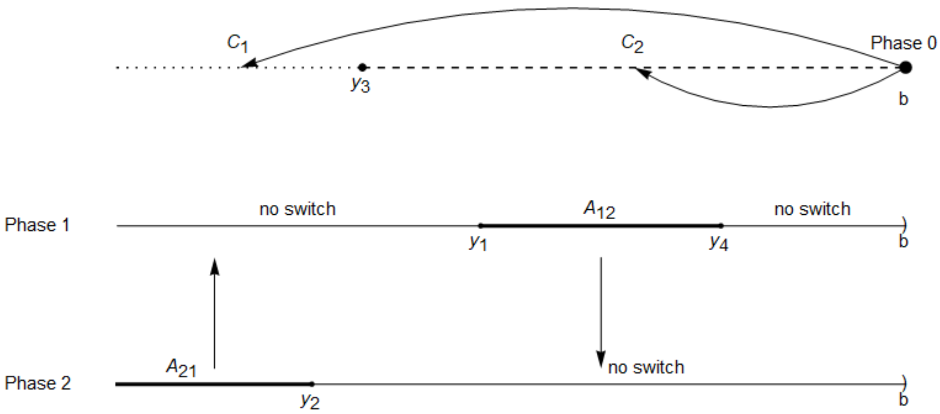

Figure 11: Optimal strategy in third example.

In Figure 12, we show the discounted total cost $V_1(x)$ (dotted), $V_2(x)$ (dashed) and $V_0(b)$ (solid point) of the optimal strategy; in Figure 13, the discounted holding cost $\mathcal{H}_1^{(q)}(x)$ (dotted), $\mathcal{H}_2^{(q)}(x)$ (dashed) and $\mathcal{H}_0^{(q)}(b)$ (solid point) of the optimal strategy; in Figure 14, the discounted penalty cost $\mathcal{S}_1^{(q)}(x)$ (dotted), $\mathcal{S}_2^{(q)}(x)$ (dashed) and $\mathcal{S}_0^{(q)}(b)$ (solid point) of the optimal strategy; and finally in Figure 15, the discounted penalty cost $\mathcal{K}_1^{(q)}(x)$ (dotted), $\mathcal{K}_2^{(q)}(x)$ (dashed) and $\mathcal{K}_0^{(q)}(b)$ (solid point) of the optimal strategy.

The observations (1), (2) and (3) of Remark 7.1 also hold for this example. In this case V_1 is not differentiable aty_4. Also note, that $\mathcal{K}_2^{(q)} - \mathcal{K}_1^{(q)}$ is equal to K_{21} in A_{21} and equal to $-K_{12}$ in A_{12}; also $\mathcal{K}_2^{(q)}(b^-) - \mathcal{K}_0^{(q)}(b) = K_{20}$ and $\mathcal{K}_1^{(q)}(b^-) - \mathcal{K}_0^{(q)}(b) = K_{10}$ because (y_4, b) is the second component of the non-action zone. As in the previous examples, $\mathcal{K}_2^{(q)}$ has a downward jump at y_2 and, in this case, $\mathcal{K}_1^{(q)}$ has a downward jump at y_4 because this point is the boundary between the switching zone $A_{12} = [y_1, y_4]$

and the non-action zone (y_4, b).

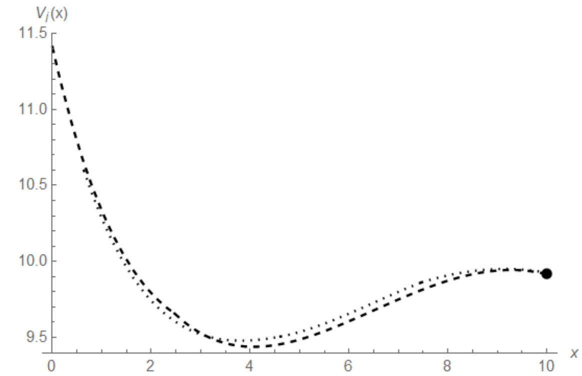

Figure 12: Third example, total cost.

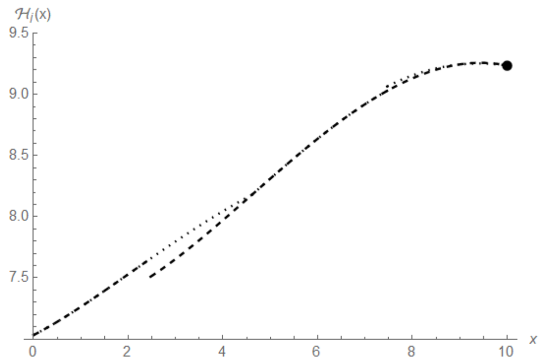

Figure 13: Third example, holding cost.

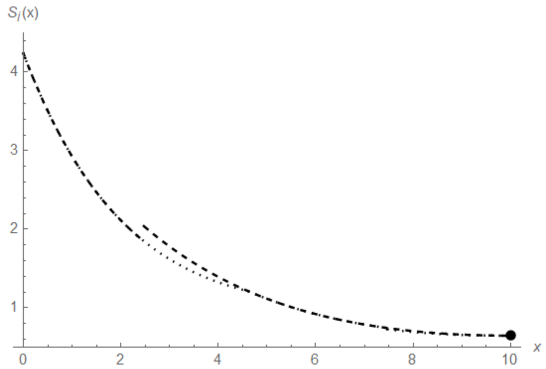

Figure 14: Third example, shortage cost.

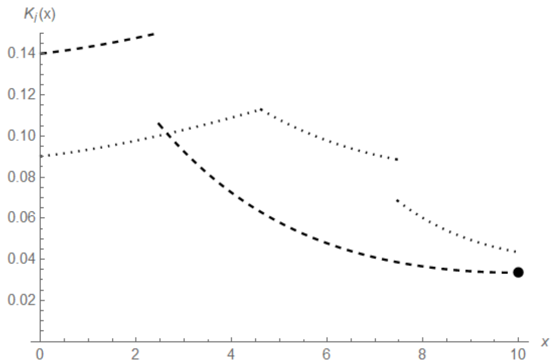

Figure 15: Third example, switching cost.

References

[1] Avram, F., Grahoavac, D., Vardar-Acre, C. (2019). The W,Z scale functons kit for first passage problems of spectrally negative Lévy processes, and applications to control problems. *https://www.esaim-ps.org/articles/ps/pdf/forth/ps170105.pdf.*

[2] Azcue P. and Muler N. (2014). *Stochastic Optimization in Insurance: a Dynamic Programming Approach.* Springer Briefs in Quantitative Finance. Springer.

[3] Azcue P. and Muler N. (2015). Optimal dividend payment and regime switching in a Compound Poisson risk model. *SIAM J. Control Optim.* 53(5), 3270-3298.

[4] Barron Y., Perry D., Stadje W. (2014). A jump-fluid production-inventory model with a double band control. *Probability in the Engineering and Informational Sciences* 28, 313333.

[5] Bayraktar E. and Egami M. (2010). On the optimal switching problem for one dimensional-diffusions. *Mathematics of Operations Research* 35(1),140-159.

[6] Brekke K. and Oksendal B. (1994). Optimal switching in an economic activity under uncertainty, *SIAM J. Cont. Optim.*, 32, 1021-1036.

[7] Crandall, M. G. and Lions, P. L. (1983). Viscosity solution of Hamilton-Jacobi equations. *Trans. Amer. Math. Soc.* 277, 1-42.

[8] De Kok, A.G., Tijms, H.C., Van Der Schouten, F.A. (1984). Approximations for the single-product production-inventory problem with compound Poisson demand and service-level constraints. *Advances in Applied Probability*, 16(2),378-401.

[9] De Kok, A.G. (1985). Approximations for a lost-sales production-inventory control model with service level constraints. *Management Sciences*, 31 (6), 729-737.

[10] De Kok, A.G. (1987). Approximations for operating characteristics in a production-inventory model with variable production rate. *European Journal of Operations research*, 29, 286-297.

[11] Doshi, B.T., Van Der Duyn Schouten, F.F. and Talman, A.J.J (1978). A production-inventory control model with a mixture of backorders and lost sales. *Management Science*, 24 (10), 1078-1086.

[12] Fleming, W.H. and Soner, H.M. (1993). *Controlled Markov Processes and Viscosity Solutions.* New York: Springer-Verlag.

[13] Gavish, B., Graves, S.C. (1980), A one product production-inventory problem under continuous review policy. *Operations Research*, 28(5), 1222-1235.

[14] Graves, S.C., Keilson, J. (1981). The compensation method applied to one-product production-inventory problems. *Mathematics of Operations Research*, 6 (2), 246-262.

[15] Hamadène S. and Jeanblanc M. (2007). On the Starting and Stopping Problem: Application in Reversible Investments. *Mathematics of Operations Research* 32(1),182-192.

[16] Kella O, Whitt W. (1992). Useful martingales for stochastic storage processes with Lévy input. *Journal of Applied Probability*, 29, 396-403.

[17] Kyprianou, A. E. (2014). *Fluctuations of Lévy Processes with Applications.* Springer Berlin Heidelberg.

[18] Pham H. and Ly Vath V. (2007). Explicit solution to an optimal switching problem in the two-regime case, *SIAM J. Cont. Optim.*, 46, 395-426.

[19] Pham, H., Ly Vath, V. and Zhou, X. Y. (2009). Optimal Switching over Multiple Regimes. *SIAM J. Control Optim.* 48, 2217–2253.

[20] Protter, P. (1992). *Stochastic integration and differential equations.* Berlin: Springer Verlag.

[21] Shi, J., Katehakis, M.H., Melamed, B.,Xia, Y. (2014). Production-inventory systems with lost sales and compound Poisson demands. *Operations Research.* 62 (5), 1048-1063.