ON THE $n-$TH LINEAR POLARIZATION CONSTANT OF $\mathbb{R}^n$.

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To my daughters, Ana and Lara.

ABSTRACT. We prove that given any set of n unit vectors $\{v_i\}_{i=1}^n \subset \mathbb{R}^n$, the inequality

$$\sup_{\|x\|_{\mathbb{R}^n}=1} |\langle x, v_1\rangle \cdots \langle x, v_n\rangle| \geq n^{-n/2}$$

holds for $n \leq 14$. Moreover, the equality is attained if and only if $\{v_i\}_{i=1}^n$ is an orthonormal system.

1. INTRODUCTION.

Let $P_1, \ldots, P_n$ be homogeneous polynomials defined on $\mathbb{R}^m, \mathbb{C}^m$ or, in general, in any Banach space. Given a norm $\|\cdot\|$ defined on the space of polynomials, the problem of finding a constant M, depending only on the degrees of $P_1, \ldots, P_n$, such that

$$\|P_1\| \cdots \|P_n\| \leq M \|P_1 \cdots P_n\|$$

has been extensively studied by many authors. In this note we are concerned with a special case: given any set $\{\phi_i\}_{i=1}^n$ of continuous linear functionals defined on a Hilbert space H, we study the inequality

$$\|\phi_1\| \cdots \|\phi_n\| \leq M \, \|\phi_1 \cdots \phi_n\|, \tag{1}$$

where $\phi_1 \cdots \phi_n$ is the $n-$homogeneous polynomial defined by the pointwise product

$$\phi_1 \cdots \phi_n(x) = \phi_1(x) \cdots \phi_n(x),$$

and $\|\cdot\|$ is the uniform norm over the unit sphere of H.

For a Banach space E with dual space E' and considering the uniform norm on the unit sphere of E, C. Benítez, Y. Sarantopoulos and A. Tonge (see [6]) defined the $n-$th linear polarization constant of E

$$\begin{aligned}\mathbf{c}_n(E) =& \inf \{M > 0 : \|\phi_1\| \cdots \|\phi_n\| \leq M \, \|\phi_1 \cdots \phi_n\|, \forall \, \phi_1, \ldots \phi_n \in E'\} \\ =& 1/\inf \left\{ \sup_{\|x\|=1} |\phi_1(x) \cdots \phi_n(x)| : \phi_i \in E', \|\phi_i\| = 1 \; \forall \; 1 \leq i \leq n \right\}.\end{aligned}$$

In [21], R. Ryan and B. Turett, studying the geometry of spaces of polynomials, showed that for each n there is a constant K_n such that $\mathbf{c}_n(E) \leq K_n$ for every Banach space E. In [6], it was proved that the best constant K_n for complex Banach spaces is n^n and S. G. Révész and Y. Sarantopoulos [20] proved that the best constant K_n for real Banach spaces is also n^n. Note that $\mathbf{c}_n(\ell_1^n) = n^n$, but in general, for different Banach spaces it is possible to find smaller values for $\mathbf{c}_n(E)$.

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In the last two decades there has been many research articles on this topic, for different techniques and approaches in calculating polarization constants see, for example, [1], [8], [9], [10], [11], [12], [13], [14], [16], [17], [18] and [20] and the references therein.

Let E be a Banach space, a *plank* or *strip* is the set of points between two parallel hyperplanes in E. Given a convex body $K \subset E$, and any norm one linear functional $\phi \in E'$, we can measure the distance between two supporting hyperplanes of K, defined by level sets of ϕ. This distance is the width of K in the direction induced by ϕ. The *minimal width of* K is the minimal width among all directions induced by norm one linear functionals. The following question was posed by A. Tarski (see [22]):

Let K be a convex body covered by n parallel planks, is it true that the sum of the widths of each plank is not less than the minimal width of K?

A positive answer was given by T. Bang in [5], who presented a strengthened version of this question considering the sum of the *relative widths* instead of the widths of the planks. This is still an open problem in the general case, but for centrally symmetric bodies a positive answer was given by K. Ball in [4]. It is worth noting that linear polarization constants are related to *plank problems* in Banach spaces, in particular the upper bound $\mathbf{c}_n(E) \leq n^n$ for any real Banach space can be deduced from Theorem 2 in [4].

In [20], using the remarkable theorem of A. Dvoretzky (see [7], [15]), it is shown that Hilbert spaces have the smallest $n-$th polarization constant among infinite dimensional Banach spaces. Namely, we have $\mathbf{c}_n(\ell_2^n) \leq \mathbf{c}_n(E)$ for any infinite dimensional Banach space E. So, knowing the exact value of $\mathbf{c}_n(\ell_2^n)$ becomes an interesting and intriguing problem. Working in a Hilbert space H, by Riesz representation theorem, inequality (1) may be written in the following way

$$\|v_1\| \cdots \|v_n\| \leq M \sup_{\|x\|_H=1} |\langle x, v_1\rangle \cdots \langle x, v_n\rangle|.$$

Note that we can modify any vector v_i to $-v_i$ at our convenience, without altering either sides of the inequality.

Given an orthonormal basis $\{e_i\}_{i=1}^n \subset \mathbb{R}^n$, by the Arithmetic-Geometric mean inequality, for any unit vector $x \in \mathbb{R}^n$ we have

$$\prod_{i=1}^n |\langle x, e_i\rangle| = \left(\prod_{i=1}^n |\langle x, e_i\rangle|^2\right)^{1/2} \leq \left(\frac{1}{n}\sum_{i=1}^n |\langle x, e_i\rangle|^2\right)^{n/2} = n^{-n/2}.$$

From this bound, it follows that $\mathbf{c}_n(\mathbb{R}^n) \geq \sqrt{n^n}$. In [6], C. Benítez, Y. Sarantopoulos and A. Tonge asked if $\mathbf{c}_n(\mathbb{R}^n) = n^{n/2}$, making the following conjecture.

Conjecture 1. *Given n unit vectors $\{v_i\}_{i=1}^n \subset \mathbb{R}^n$, then*

$$\sup_{\|x\|_{\mathbb{R}^n}=1} |\langle x, v_1\rangle \cdots \langle x, v_n\rangle| \geq n^{-n/2}, \tag{2}$$

and equality holds if and only if $\{v_i\}_{i=1}^n$ is an orthonormal system.

For $n \leq 5$, the inequality was proved by A. Pappas and S. G. Révész in [18] (see also [17]). However, the question remained unanswered for $n \geq 6$.

A complex analogue of inequality (2) was proved by J. Arias-de-Reyna in [2]. More precisely, the author showed that for any set of unit vectors $\{z_i\}_{i=1}^n \subset \mathbb{C}^n$ it follows that

$$\sup_{\|z\|_{\mathbb{C}^n}=1} |\langle z, z_1\rangle \cdots \langle z, z_n\rangle| \geq n^{-n/2}. \tag{3}$$

In [3], K. Ball proved a stronger result, which is known as "*the complex plank problem for Hilbert spaces*" and also implies inequality (3). Finally, in [19], the author showed that a set $\{z_i\}_{i=1}^n$ of unit vectors in a complex Hilbert space H for which the equality is attained must be an orthonormal system.

Although the exact value of the $n-$th linear polarization constant of $\mathbb{R}^n$ is not known for all $n \in \mathbb{N}$, there are several articles in the literature finding upper bounds for its value. In fact, the existence of $c \in \mathbb{R}$ such that $\mathbf{c}_n(\mathbb{R}^n) \leq (cn)^{n/2}$ was studied by many authors: A. E. Litvak, V. D. Milman, and G. Schechtman [11] for $c \approx 12,67$, J. C. García-Vázquez and R. Villa [9] with $c \approx 3,57$, S. G. Révész and Y. Sarantopoulos [20] proved that $\mathbf{c}_n(\mathbb{R}^n) \leq 2^{n/2-1}\, n^{n/2}$, P. E. Frenkel [8] improved the previous bound showing that $\mathbf{c}_n(\mathbb{R}^n) \leq \left(3^{3/2}e^{-1}\right)^{n/2} n^{n/2}$, and G. A. Muñoz-Fernández et al. [16] showed that $\mathbf{c}_n(\mathbb{R}^n) \leq n2^{n/4}n^{n/2}$, which is asymptotically tighter than the previous bounds.

The aim of this work is to extend the validity range of Conjecture 1. In Section 2, given $n \in \mathbb{N}$, we study the minima of some constrained problems depending on a parameter s. Namely, we have sets $\Sigma_s \subset \mathbb{R}^n$, a function $f : \Sigma_s \to \mathbb{R}$, and compute $\min\limits_{a\in\Sigma_s} f(a)$, and then we find the minima of $s \mapsto \min\limits_{a\in\Sigma_s} f(a)$. Finally, in Section 3, we will apply the results from Section 2 to prove, for $6 \leq n \leq 14$, that the $n-$th linear polarization constant of $\mathbb{R}^n$ is $n^{n/2}$. Moreover, we show that we will have an equality in (2) if and only if $\{v_i\}_{i=1}^n$ is an orthonormal system.

2. Some Useful Inequalities and Constrained Problems.

In this section, for our purposes, we need to present and prove some useful inequalities. We will consider $n \in \mathbb{N}, n \geq 2, s \in [\sqrt{n}, n]$ and $Q_s = [s^{-1}, 1]^n$. Given the function $f : Q_s \to \mathbb{R}$, defined by $f(a) = a_1 \cdots a_n$, we are interested in finding the constrained minima

$$\min_{a\in Q_s} f(a), \text{ subject to } \sum_{i=1}^n a_i = s.$$

Let us denote by Σ_s the set $Q_s \cap \{a \in \mathbb{R}^n : \sum_{i=1}^n a_i = s\}$, and define the function $\mu : [\sqrt{n}, n] \to \mathbb{R}$, by $\mu(s) = \min\limits_{a\in\Sigma_s} f(a)$.

It is easy to see, applying Lagrange multipliers, that

$$\max_{a\in\Sigma_s} f(a) = f\left(\frac{s}{n}, \frac{s}{n}, \dots, \frac{s}{n}\right) = \left(\frac{s}{n}\right)^n.$$

So, we might suspect that $\mu(s)$ should be reached on the intersection of the hyperplane and a face of the cube. Moreover, it is reasonable to think that the function f gets smaller as more coordinates of a take the value s^{-1}. Following this idea, let us define

$$k_0(s) = \min\left\{k \in \mathbb{N} : n - k < s - ks^{-1}\right\}.$$

It is clear that $1 \leq k_0(s) \leq n+1$, and if $k_0(s) < n$, this value gives us the first coordinate k, such that any point $a \in Q_s$,

$$a = (\underbrace{s^{-1}, s^{-1}, \dots, s^{-1}}_{k_0(s)-times}, a_{k_0(s)+1}, \dots, a_n)$$

does not belong to Σ_s.

Remark 2.1. Note that from the very definition of $k_0(s)$, we have

$$n - k_0(s) < s - k_0(s)s^{-1}, \tag{4}$$

which is equivalent to

$$\frac{s(n-s)}{s-1} < k_0(s).$$

Also, since

$$s - (k_0(s)-1)s^{-1} \leq n + 1 - k_0(s), \tag{5}$$

we obtain

$$k_0(s) \leq \frac{s(n-s)}{s-1} + 1.$$

As usual, if $\lfloor x \rfloor = \max\{m \in \mathbb{Z} : m \leq x\}$ denotes the floor function, we can write

$$k_0(s) = \left\lfloor \frac{s(n-s)}{s-1} \right\rfloor + 1.$$

Proposition 2.2. *Let $f : Q_s \to \mathbb{R}$ be defined by*

$$f(a_1, a_2, \ldots, a_n) = a_1 a_2 \cdots a_n.$$

Then,

$$\mu(s) = s^{1-k_0(s)} \left(s - (k_0(s)-1)s^{-1} - n + k_0(s) \right).$$

Proof. By continuity of f and compactness of Σ_s we know that the minimum is attained at some point $a \in \Sigma_s$. Since f is a symmetric function, we may assume that

$$s^{-1} \leq a_1 \leq a_2 \leq \ldots \leq a_n \leq 1.$$

First, let us show that $a_{k_0(s)-1} = s^{-1}$. If not, from the definition of $k_0(s)$, we have that $n - (k_0(s)-1) \geq s - (k_0(s)-1)s^{-1}$. Then, we obtain

$$\begin{aligned} s = \sum_{i=1}^{n} a_i &> (k_0(s)-1)s^{-1} + (n+1-k_0(s))\, a_{k_0(s)} \\ &\geq s - n - 1 + k_0(s) + (n+1-k_0(s))\, a_{k_0(s)} \\ &= s + (n+1-k_0(s)) \left(a_{k_0(s)} - 1 \right). \end{aligned}$$

This inequality implies that $a_{k_0(s)} < 1$. Now, taking

$$\varepsilon \in \left(0, \min\left(a_{k_0(s)-1} - s^{-1}, 1 - a_{k_0(s)}\right)\right),$$

we slightly perturb the values $a_{k_0(s)-1}$ and $a_{k_0(s)}$ by ε:

$$\begin{aligned} a_1 \cdots a_n &\leq a_1 \cdots \left(a_{k_0(s)-1} - \varepsilon\right)\left(a_{k_0(s)} + \varepsilon\right) \cdots a_n \\ &= a_1 \cdots \left(a_{k_0(s)-1} a_{k_0(s)} + \varepsilon \left(a_{k_0(s)-1} - a_{k_0(s)}\right) - \varepsilon^2\right) \cdots a_n \\ &< a_1 \cdots a_n, \end{aligned}$$

which is impossible. Then, $a_{k_0(s)-1} = s^{-1}$.

Our next step is to find the minimum of $g : [s^{-1}, 1]^{n+1-k_0(s)} \to \mathbb{R}$, defined by

$$g(a_{k_0(s)}, \ldots, a_n) := f(s^{-1}, s^{-1}, \ldots, s^{-1}, a_{k_0(s)}, \ldots, a_n),$$

subject to the equality constraint

$$\sum_{i=k_0(s)}^{n} a_i = s - (k_0(s)-1)s^{-1} \in \left(n - k_0(s) + s^{-1}, n - k_0(s) + 1\right).$$

Write $\widetilde{s} = s - (k_0(s)-1)s^{-1}$. Let us prove that the minimum is attained at

$$(a_{k_0(s)}, \ldots, a_n) = (\widetilde{s} - n + k_0(s), 1, \ldots, 1).$$

Note that if $a_{k_0(s)} > \widetilde{s} - n + k_0(s)$, then it must be $a_{k_0(s)+1} < 1$, and we can proceed as we did in the beginning: i.e. choosing $\varepsilon > 0$, small enough, we can see that

$$\begin{aligned} a_{k_0(s)} a_{k_0(s)+1} \cdots a_n &\leq \left(a_{k_0(s)} - \varepsilon\right)\left(a_{k_0(s)+1} + \varepsilon\right) \cdots a_n \\ &< a_{k_0(s)} a_{k_0(s)+1} \cdots a_n, \end{aligned}$$

which leads to a contradiction. Therefore, $a_{k_0(s)} = \widetilde{s} - n + k_0(s)$, and $a_i = 1$ for $k_0(s) + 1 \leq i \leq n$. It follows immediately that

$$\mu(s) = s^{1-k_0(s)}\left(s - (k_0(s)-1)s^{-1} - n + k_0(s)\right). \qquad \square$$

Corollary 2.3. *Let $f : Q_s \to \mathbb{R}$ be the function defined by $f(a_1, a_2, \ldots, a_n) = a_1 a_2 \cdots a_n$. Then,*

$$\mu(s) = \frac{s^{-1} + s - n + \left(\left\lfloor \frac{s(n-s)}{s-1} \right\rfloor + 1\right)(1 - s^{-1})}{s^{\left\lfloor \frac{s(n-s)}{s-1} \right\rfloor}}.$$

Proof. The proof is immediate by combining Remark 2.1 and Proposition 2.2. $\square$

Remark 2.4. Examining inequalities (4) and (5), we deduce that

$$n - k_0(s) < s - k_0(s)s^{-1} \leq n + 1 - k_0(s) - s^{-1} < n + 1 - k_0(s).$$

Then, we obtain

$$n - k_0(s) = \left\lfloor s - k_0(s)s^{-1} \right\rfloor. \tag{6}$$

Remark 2.5. The function μ is continuous for $s \in [\sqrt{n}, n]$, except for those points $\{s_j\}_{j=0}^n$ where $\frac{s_j(n-s_j)}{s_j-1} = j$. We can compute s_j as the positive root of the equation $x^2 - (n-j)x - j = 0$, namely

$$s_j = \frac{(n-j) + \sqrt{(n-j)^2 + 4j}}{2}.$$

Also, since $s \mapsto \frac{s(n-s)}{s-1}$ is a decreasing function, we have

$$\sqrt{n} = s_n < s_{n-1} < \ldots < s_1 < s_0 = n.$$

Proposition 2.6. *The function $\mu : [\sqrt{n}, n] \to \mathbb{R}$ is lower semi-continuous.*

Proof. As we noted in Remark 2.5, the function μ is continuous on its domain, except at the points $\{s_j\}_{j=0}^n$. Then, it remains to show that $\liminf_{s \to s_j} \mu(s) \geq \mu(s_j)$ for $0 \leq j \leq n$. However, since $s \mapsto \frac{s(n-s)}{s-1}$ decreases, we know that $s \mapsto \left\lfloor \frac{s(n-s)}{s-1} \right\rfloor$ is continuous from the left, so it is not necessary to study the case $j = 0$ and we only have to check that $\liminf_{s \to s_j^+} \mu(s) \geq \mu(s_j)$ for $1 \leq j \leq n$.

First, let us compute $\mu(s_j)$. We can write, from Corollary 2.3,

$$\begin{aligned}\mu(s_j) &= \frac{s_j^{-1} + s_j - n + \left(\left\lfloor \frac{s_j(n-s_j)}{s_j-1} \right\rfloor + 1\right)(1 - s_j^{-1})}{s_j^{\left\lfloor \frac{s_j(n-s_j)}{s_j-1} \right\rfloor}} \\ &= \frac{s_j^{-1} + s_j - n + (j+1)\,(1 - s_j^{-1})}{s_j^{j}} \\ &= \frac{1 + s_j^2 - (n-j-1)s_j - (j+1)}{s_j^{j+1}} \\ &= \frac{s_j^2 - (n-j)s_j - j + s_j}{s_j^{j+1}} = \frac{1}{s_j^j},\end{aligned}$$

where in the last step we have used that $s_j^2 - (n-j)s_j - j = 0$.

For $s > s_j$, close enough, $\left\lfloor \frac{s(n-s)}{s-1} \right\rfloor + 1 = k_0(s) = k_0(s_j) - 1 = j$. Then, combining with (6), we have

$$\begin{aligned}\liminf_{s \to s_j^+} \mu(s) &= \liminf_{s \to s_j^+} \frac{s^{-1} + s - js^{-1} - \lfloor s - js^{-1} \rfloor}{s^{j-1}} \\ &\geq \liminf_{s \to s_j^+} \frac{s^{-1}}{s^{j-1}} = \frac{1}{s_j^j} = \mu(s_j).\end{aligned}$$

□

The following lemma is crucial to determine the minimum of the function μ.

Lemma 2.7. *Given $j, n \in \mathbb{N}$, such that $n \geq 2$ and $1 \leq j \leq n$, then the function $M_j : [s_j, s_{j-1}] \to \mathbb{R}$, defined by*

$$M_j(x) = x^{2-j} + (j-n)x^{1-j} + (1-j)x^{-j}$$

is a quasi-concave function.

Proof. First, note that for $j = 1$ and $j = 2$, the function M_j is concave and the lemma follows. For the remaining cases, when $3 \leq j \leq n$, we will prove that M_j satisfies one of the following conditions:

- M_j is an increasing function on $[s_j, s_{j-1}]$.
- There exists $t_j \in (s_j, s_{j-1})$ such that M_j is an increasing function on $[s_j, t_j]$ and it is a decreasing function on $[t_j, s_{j-1}]$.

Let us compute

$$\begin{aligned}M_j'(x) &= (2-j)x^{1-j} + (j-n)(1-j)x^{-j} - j(1-j)x^{-j-1} \\ &= \frac{(2-j)x^2 + (j-n)(1-j)x - j(1-j)}{x^{j+1}} \\ &= \frac{x^2 + (1-j)\left(x^2 + (j-n)x - j\right)}{x^{j+1}}.\end{aligned}$$

Then, to determine the behaviour of M_j it will be enough to analyze the sign of the concave quadratic function

$$x \mapsto x^2 + (1-j)\left(x^2 + (j-n)x - j\right).$$

By definition, $s_j^2 + (j-n)s_j - j = 0$. We have $M_j'(s_j) > 0$, then M_j has at most one critical point in the interval $[s_j, s_{j-1}]$, and the assertion is proved. □

Finally, we need the following proposition in order to prove the main theorem of this section.

Proposition 2.8. *Given* $n \in \mathbb{N},\ 2 \le n \le 14,$ *and* $0 \le j \le n,$ *then* $s_j^j \le \sqrt{n^n}$.

Proof. From Remark 2.5 we know that

$$\sqrt{n} = s_n < s_{n-1} < \ldots < s_1 < s_0 = n,$$

then

$$\left(\sqrt{n}\right)^{n/2} = (s_n)^{n/2} < (s_{n-1})^{n/2} < \ldots < (s_1)^{n/2} < (s_0)^{n/2} = n^{n/2}.$$

The last inequalities show that $s_j^j \le \sqrt{n^n}$ for $j \le \lfloor n/2 \rfloor$. Then, we can restrict ourself to $3 \le n \le 14$, and $\lfloor n/2 \rfloor + 1 \le j \le n-1$. Note that, for such values of j,

$$\sqrt{n} < n^{\frac{n}{2(n-1)}} \le n^{\frac{n}{2j}} \le n^{\frac{n}{2(\lfloor n/2 \rfloor + 1)}} < n.$$

We will prove that

$$\left(n^{n/2j}\right)^2 - (n-j)n^{n/2j} - j \ge 0, \tag{7}$$

which is equivalent to $s_j \le n^{n/2j}$. Let us write inequality (7) as follows

$$\left(n^{n/2j}\right)^2 - nn^{n/2j} \ge (1 - n^{n/2j})j. \tag{8}$$

If we call $x = n^{n/2j}$, then we have $j = \frac{\ln\left(\sqrt{n^n}\right)}{\ln(x)}$, and inequality (8) becomes

$$\frac{1-x}{x^2 - nx} - \frac{\ln(x)}{\ln\left(\sqrt{n^n}\right)} \ge 0 \quad \text{for } x \in J_n = \left[n^{\frac{n}{2(n-1)}}, n^{\frac{n}{2(\lfloor n/2 \rfloor + 1)}}\right]. \tag{9}$$

Note that $J_n \subset (\sqrt{n}, n)$ and, in order to prove inequality (9), we may define the function $\phi : (0, n) \to \mathbb{R}$ by

$$\phi(x) = \frac{1-x}{x^2 - nx} - \frac{\ln(x)}{\ln\left(\sqrt{n^n}\right)},$$

and show that $\phi\left(n^{\frac{n}{2(n-1)}}\right) \ge 0$ and that $\phi(x)$ is an increasing function on J_n. For this purpose we can study

$$\phi'(x) = \frac{x^2 - 2x + n}{x^2(x-n)^2} - \frac{1}{x \ln\left(\sqrt{n^n}\right)}.$$

To prove that $\phi'(x) > 0$, we may write

$$\phi'(x) = \frac{1}{x \ln\left(\sqrt{n^n}\right)} \left(\ln\left(\sqrt{n^n}\right) \ \frac{x^2 - 2x + n}{x(x-n)^2} - 1 \right),$$

and prove that

$$\ln\left(\sqrt{n^n}\right) \ \frac{x^2 - 2x + n}{x(x-n)^2} > 1 \quad \text{for all } x \in J_n.$$

Let us analyze the function $\varphi(x) = \dfrac{x^2 - 2x + n}{x(x-n)^2}$. Its derivative is just

$$\varphi'(x) = \frac{x^3 + (n-4)\,x^2 + 3nx - n^2}{x^2\,(n-x)^3}.$$

The sign of $\varphi'(x)$ over the interval J_n depends on the sign of the function

$$x \mapsto x^3 + (n-4)\,x^2 + 3nx - n^2.$$

But this is a monotone increasing function (for $2 \leq n \leq 14$) because its derivative, $3x^2+2(n-4)x+3n$, has no real roots and is positive on $\mathbb{R}$. In fact, the discriminant $\Delta = 4(n^2-17n+16)$ could be factored as $\Delta = 4(n-1)(n-16)$, which is negative for $2 \leq n \leq 15$. Then, for $x \in [\sqrt{n}, n)$ we obtain

$$\begin{aligned} \operatorname{sign}\left(\varphi'(x)\right) &= \operatorname{sign}\left(\sqrt{n}^3 + (n-4)\,n + 3n\sqrt{n} - n^2\right) \\ &= \operatorname{sign}\left(4\left(\sqrt{n^3} - n\right)\right) = 1. \end{aligned}$$

Since φ is an increasing function on J_n, we may show that $\ln\left(\sqrt{n^n}\right)\varphi\left(n^{\frac{n}{2(n-1)}}\right) > 1$, i.e.

$$\ln\left(\sqrt{n^n}\right)\ \frac{\left(n^{\frac{n}{2(n-1)}}\right)^2 - 2\left(n^{\frac{n}{2(n-1)}}\right) + n}{\left(n^{\frac{n}{2(n-1)}}\right)\left(\left(n^{\frac{n}{2(n-1)}}\right) - n\right)^2} > 1,$$

to conclude that $\phi'(x) > 0$ on J_n. Once this is done, it remains to check that $\phi\left(n^{\frac{n}{2(n-1)}}\right) > 0$ to ensure that the inequality (9) is satisfied. But, as we said, this is equivalent to show that $s_{n-1}^{n-1} \leq \sqrt{n^n}$.

The following table contains these values for $3 \leq n \leq 16$.

n	$\ln(\sqrt{n^n})\ \frac{\left(n^{\frac{n}{2(n-1)}}\right)^2 - 2\left(n^{\frac{n}{2(n-1)}}\right) + n}{\left(n^{\frac{n}{2(n-1)}}\right)\left(\left(n^{\frac{n}{2(n-1)}}\right) - n\right)^2}$	S_{n-1}^{n-1}	$\sqrt{n^n}$
3	$\approx 5,065$	4	$\approx 5,196$
4	$\approx 2,666$	$\approx 12,211$	16
5	$\approx 2,008$	$\approx 43,053$	$\approx 55,901$
6	$\approx 1,698$	$\approx 169,442$	216
7	$\approx 1,514$	729	$\approx 907,492$
8	$\approx 1,389$	$\approx 3380,607$	4096
9	$\approx 1,298$	$\approx 16725,933$	19683
10	$\approx 1,227$	$\approx 87610,098$	100000
11	$\approx 1,170$	$\approx 482892,455$	$\approx 534145,739$
12	$\approx 1,123$	$\approx 2787117,027$	2985984
13	$\approx 1,084$	16777216	$\approx 17403307,350$
14	$\approx 1,049$	$\approx 104973424,100$	105413504
15	$\approx 1,019$	$\approx 680750436,468$	$\approx 661735513,918$
16	$\approx 0,992$	4564290812,351	4294967296

The second columm shows that, for $3 \leq n \leq 15$, $\ln\left(\sqrt{n^n}\right)\ \varphi\left(n^{\frac{n}{2(n-1)}}\right)$ is greater than 1. Comparing the third with the fourth column, we see that s_{n-1}^{n-1} is less than or equal to $\sqrt{n^n}$ for $3 \leq n \leq 14$. As both inequalities are fulfilled for $3 \leq n \leq 14$, the assertion is proved. $\square$

Remark 2.9. Note that for $n = 15$ and $n = 16$ we have $s_{n-1}^{n-1} > \sqrt{n^n}$. Moreover, for $n = 16$ we can have $\phi'(x) < 0$ at some points.

Remark 2.10. Actually we have proved that $s_j^j < \sqrt{n^n}$, unless $j = n$.

Theorem 2.11. *Given $n \in \mathbb{N}$, $2 \leq n \leq 14$, let $f : Q_s \to \mathbb{R}$ be the function defined by $f(a_1, a_2, \ldots, a_n) = a_1 a_2 \cdots a_n$. If we consider $\mu : [\sqrt{n}, n] \to \mathbb{R}$, where $\mu(s) = \min_{a \in \Sigma_s} f(a)$, then*

$$\mu(s) \geq \frac{1}{\sqrt{n^n}}.$$

Moreover, the minimum is attained only at $s = \sqrt{n}$.

Proof. Let us begin by restricting μ to the open interval $I_j = (s_j, s_{j-1})$, for $1 \leq j \leq n$. From Corollary 2.3 we can write

$$\mu(s) = \frac{s^{-1} + s - n + \left(\left\lfloor \frac{s(n-s)}{s-1} \right\rfloor + 1\right)(1 - s^{-1})}{s^{\left\lfloor \frac{s(n-s)}{s-1} \right\rfloor}}.$$

Since $\left\lfloor \frac{s(n-s)}{s-1} \right\rfloor + 1 = j$ for any $s \in I_j$, we obtain

$$(\mu_{|I_j})(s) = \frac{s^{-1} + s - n + j(1 - s^{-1})}{s^{j-1}} = s^{2-j} + (j-n)s^{1-j} + (1-j)s^{-j}.$$

Recall that μ is a lower semi-continuous function, and it must attain its minimum on any compact set. Therefore, there exists some point $\mathbf{s_j} \in [s_j, s_{j-1}]$, such that $\mu(s) \geq \mu(\mathbf{s_j})$ for all $s \in [s_j, s_{j-1}]$.

For any $s \in (s_j, s_{j-1})$, the evaluation of $\mu(s)$ coincides with the evaluation of the function $M_j : [s_j, s_{j-1}] \to \mathbb{R}$, considered in Lemma 2.7. Then it follows that $\mathbf{s_j}$ does not belong to the open interval (s_j, s_{j-1}).

Since

$$\min_{s \in [\sqrt{n}, n]} \mu(s) = \min_{1 \leq j \leq n} \min_{s \in [s_j, s_{j-1}]} \mu(s),$$

the minimum of $\mu(s) : [\sqrt{n}, n] \to \mathbb{R}$ must be attained at $\mathbf{s} \in \{s_j\}_{j=0}^n$, and it suffices to prove

$$\mu(s_j) \geq \frac{1}{\sqrt{n^n}}$$

for $0 \leq j \leq n$. Then, the proof follows from Proposition 2.8 and Remark 2.10. □

3. The $n-$th linear polarization constant of $\mathbb{R}^n$.

Let us begin by recalling that in order to show the equality $\mathbf{c}_n(\mathbb{R}^n) = \sqrt{n^n}$, it is enough to prove that for any set of unit vectors $\{v_i\}_{i=1}^n \subset \mathbb{R}^n$, there exists a norm one vector $x \in \mathbb{R}^n$ such that

$$|\langle x, v_1 \rangle \cdots \langle x, v_n \rangle| \geq n^{-n/2}. \tag{10}$$

In [18], the authors ensure the existence of a norm one vector $x \in \mathbb{R}^n$ satisfying inequality (10) for $n = 2, 3, 4$ and 5. The proof is based on an appropriate choice of signs $\{\varepsilon_i\}_{i=1}^n$ such that maximizes the euclidean norm of $\sum_{i=1}^n \varepsilon_i v_i$. Then, the desired vector is

$$x = \frac{\sum_{i=1}^n \varepsilon_i v_i}{\|\sum_{i=1}^n \varepsilon_i v_i\|}.$$

Note that for any choice of signs we have

$$\left\|\sum_{i=1}^{n} \varepsilon_i v_i\right\|_2^2 = \sum_{i=1}^{n} \|\varepsilon_i v_i\|_2^2 + 2 \sum_{1\le i\neq j\le n} \varepsilon_i \varepsilon_j \langle v_i, v_j\rangle.$$

If we consider the random vector of signs $(\varepsilon)_j = (\varepsilon_{j_1}, \ldots, \varepsilon_{j_n})$, all with equal probability of being chosen, the mean of the squared norm is

$$\frac{1}{2^n} \sum_{j=1}^{2^n} \left\|\sum_{i=1}^{n} \varepsilon_{j_i} v_i\right\|_2^2 = n.$$

For our purposes we may assume that the choice of signs maximizing $\|\sum_{i=1}^{n} \varepsilon_i v_i\|_2^2$ is just $(\varepsilon)_j = (1, \ldots, 1)$. In the sequel we will consider sets of unit vectors $\{v_i\}_{i=1}^{n}$ such that the *longest sum* of them is $v = \sum_{i=1}^{n} v_i$. Of course, it satisfies

$$\sqrt{n} \le \|v\| \le n.$$

For this vector v, we have $\langle v, v\rangle \ge \langle v - 2v_i, v - 2v_i\rangle$ for all $1 \le i \le n$. Then,

$$\|v\|^2 \ge \|v\|^2 - 4\langle v_i, v\rangle + 4.$$

It follows that $\langle v_i, v\rangle \ge 1$ for $1 \le i \le n$ (see [18] for further details).

Although the following result is known for $2 \le n \le 5$ (see [18]), we include these cases in the statement of our main theorem.

Theorem 3.1. *Given* $2 \le n \le 14$, *then* $\mathbf{c}_n(\mathbb{R}^n) = \sqrt{n^n}$.

Proof. Take any set of unit vectors $\{v_i\}_{i=1}^{n} \subset \mathbb{R}^n$, such that $v = \sum_{i=1}^{n} v_i$ is the *longest sum* of them. Let us show that

$$\prod_{i=1}^{n} \left\langle v_i, \frac{v}{\|v\|} \right\rangle \ge \frac{1}{\sqrt{n^n}}.$$

Write $\langle v_i, v\rangle = a_i\|v\| \ge 1$, for some $a_i \in \mathbb{R}$. Then $a_i \ge \|v\|^{-1}$ and, from Cauchy-Schwarz inequality, $a_i \le \|v_i\| = 1$. Also,

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} \left\langle v_i, \frac{v}{\|v\|} \right\rangle = \left\langle v, \frac{v}{\|v\|} \right\rangle = \|v\| \in \left[\sqrt{n}, n\right].$$

Now, applying Theorem 2.11 for $s = \|v\|$, we obtain

$$\prod_{i=1}^{n} \left\langle v_i, \frac{v}{\|v\|} \right\rangle = f(a_1, \ldots, a_n) \ge \mu(\|v\|) \ge \frac{1}{\sqrt{n^n}}. \qquad \square$$

Lemma 3.2. *Let* $\{v_i\}_{i=1}^{n} \subset \mathbb{R}^n$ *be unit vectors such that for any choice of signs* ε_i, *we have* $\|\sum_{i=1}^{n} \varepsilon_i v_i\|^2 = n$, *then* $\{v_i\}_{i=1}^{n}$ *is an orthonormal system.*

Proof. Let us call $\mathfrak{I} = \{1, 2, \ldots, n\}$. For $j \in \mathfrak{I}$, let $\mathfrak{I}_j$ be the set $\mathfrak{I} - \{j\}$. Since

$$\left\|\sum_{i\in\mathfrak{I}_j} \varepsilon_i v_i + v_j\right\|^2 = \left\|\sum_{i\in\mathfrak{I}_j} \varepsilon_i v_i - v_j\right\|^2$$

it follows that

$$\left\langle \sum_{i\in\mathfrak{I}_j} \varepsilon_i v_i, v_j \right\rangle = 0$$

for any choice of signs $\{\varepsilon_i\}_{i\in\mathfrak{I}_j} \subset \{-1,1\}^{n-1}$. Finally, since

$$\operatorname{span}\left\{\sum_{i\in\mathfrak{I}_j}\varepsilon_i v_i : \varepsilon_i \in \{-1,1\}\right\} = \operatorname{span}\{v_i : i \in \mathfrak{I}_j\},$$

we deduce that $\langle v_i, v_j\rangle = 0$ for $i \in \mathfrak{I}_j$. Now, since we can freely choose $j \in \mathfrak{I}$, the lemma is proved. □

Theorem 3.3. *For $2 \le n \le 14$, if $\{v_i\}_{i=1}^n \subset \mathbb{R}$ are unit vectors such that*

$$\sup_{\|x\|=1} |\langle x, v_1\rangle \cdots \langle x, v_n\rangle| = n^{-n/2},$$

then $\{v_i\}_{i=1}^n$ is an orthonormal system.

Proof. Recall, from Remark 2.10, that the value $\sqrt{n^n}$ is attained only if $s = \sqrt{n}$. Now, the *longest sum* $v = v_1 + \ldots + v_n$ must have norm $\sqrt{n}$. But then, any vector $v_{(\varepsilon)_j} = \sum_{i=1}^n \varepsilon_{j_i} v_i$ has norm $\sqrt{n}$. Hence, the assertion follows from the previous lemma. □

Final Remark. From [18] we knew that the *longest sum* of a set of vectors was a good candidate to check if inequality (2) holds. In this work we extended from $n = 5$ to $n = 14$ the validity range of Conjecture 1, by applying the results from Section 2. For $n = 34$, M. Matolcsi and G. A. Muñoz (see [14]) gave an example where the *longest sum* v of some set of unit vectors $\{v_i\}_{i=1}^{34}$ does not satisfy the inequality $\prod |\langle v_i, v\rangle| \ge 34^{-17}$. However, the inequality holds in some alternative vector. From their example it is possible to construct many others for any $n > 34$.

Given $s \in [\sqrt{n}, n]$, if we denote by $\mathcal{F}(s)$ the set of all $n-$tuples of unit vectors $\{v_i\}_{i=1}^n$, such that its *longest sum* v has $\|v\| = s$, then the map

$$\Lambda : \mathcal{F}(s) \longrightarrow \Sigma_s$$

defined by

$$\Lambda(v_1, v_2, \ldots, v_n) = \frac{1}{s}\left(\langle v_1, v\rangle, \ldots, \langle v_n, v\rangle\right)$$

is not necessarily surjective. Then it is possible that the *longest sum* v still works as a good tester for Inequality (2) for some other values of $14 < n < 34$.

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