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ORIGINAL PAPER

Farkas' Lemma in the bilinear setting and evaluation functionals

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Abstract
We prove the following Farkas' Lemma for simultaneously diagonalizable bilinear forms: If $A_1, \ldots, A_k$, and $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ are bilinear forms, then one—and only one—of the following holds:

(i) $B = a_1 A_1 + \cdots + a_k A_k$, with non-negative a_i's,
(ii) there exists (x, y) for which $A_1(x, y) \geq 0, \ldots, A_k(x, y) \geq 0$ and $B(x, y) < 0$.

We study evaluation maps over the space of bilinear forms and consequently construct examples in which Farkas' Lemma fails in the bilinear setting.

Keywords Farkas' Lemma · Bilinear forms · Evaluation maps

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Introduction

There are many results relating the zero sets of linear or multilinear forms and their linear dependence. For example, it is well-known that if $f_1, \ldots, f_k$ and g are linear forms such

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that

$$f_1(x) = 0, \ldots, f_k(x) = 0 \text{ imply } g(x) = 0, \tag{1}$$

then $g = a_1 f_1 + \cdots + a_k f_k$ for some scalars a_1, ..., a_k.

Also, if

$$f_1(x) \geq 0, \ldots, f_k(x) \geq 0 \text{ imply } g(x) \geq 0, \tag{2}$$

then $g = a_1 f_1 + \cdots + a_k f_k$, with $a_i \geq 0$ for all i. This is known as Farkas' Lemma [4]. Note that condition (2) above is stronger than condition (1), as can be checked by replacing x with $-x$.

Published in 1902, Farkas' Lemma leads to linear programming duality and has proved to be a central result in the development of linear and non-linear mathematical optimization. According to MathSciNet, over 250 papers cite Farkas' Lemma (one such contains the particularly beautiful statement by J. Franklin [7] that *"I hope to convince you that every mathematician should know the Farkas theorem and should know how to use it"*). One good, fairly recent source for such papers can be found among the opening articles in [5]. However, very few of these papers deal with non-linear versions of the lemma which is our interest here. In the non-linear setting, a short article describing some open problems involving polynomial matrices is contained in the work of A. A. ten Dam and J. W. Nieuwenhuis [2].

In the multilinear setting, positive results are known only in the case of two m-linear forms A and B as follows:

$$\text{If } A(x_1, \ldots, x_m) = 0 \text{ implies } B(x_1, \ldots, x_m) = 0,$$

then $B = aA$ [1].

In addition, the following weak form of Farkas' Lemma is given in [3]:

$$\text{If } A(x_1, \ldots, x_m) \geq 0 \text{ implies } B(x_1, \ldots, x_m) \geq 0,$$

then $B = aA$, with $a \geq 0$.

On the other hand, in [1] the authors give an example of symmetric bilinear forms A_1, A_2 and B such that

$$A_1(x, y) = 0 \text{ and } A_2(x, y) = 0 \text{ imply } B(x, y) = 0, \text{ but } B \neq a_1 A_1 + a_2 A_2,$$

and in [3] the author shows (non-symmetric) bilinear forms A_1, A_2 and B such that

$$A_1(x, y) \geq 0 \text{ and } A_2(x, y) \geq 0 \text{ imply } B(x, y) \geq 0, \text{ but } B \neq a_1 A_1 + a_2 A_2,$$

with non-negative a_i's.

Farkas' Lemma also fails for 2-homogeneous polynomials. Consider $P : \mathbb{R}^2 \to \mathbb{R}$ and $Q : \mathbb{R}^2 \to \mathbb{R}$ given by $P(x, y) = x^2$ and $Q(x, y) = y^2$. Then $P(x, y) \geq 0$ implies $Q(x, y) \geq 0$, since both are always non-negative, but P and Q are independent.

In view of these examples, there appears to be little room for positive results. However in Sect. 1, below, we give a version of Farkas' Lemma for simultaneously diagonalizable bilinear forms (see Theorem 1) as well as a result analogous to (1) above (see Theorem 2).

We note that a similar, but weaker, result:

$$\begin{aligned} &\text{If } A_1(x, y) \geq 0, \ldots, A_k(x, y) \geq 0 \text{ imply } B(x, y) \geq 0, \\ &\quad \text{then } B \geq a_1 A_1 + \cdots + a_k A_k \text{ with all } a_i \geq 0, \end{aligned} \tag{3}$$

(where $C \succeq D$ means $C - D$ is positive semidefinite), is known and can be obtained through the S-procedure as in [6]. We require, however, equality in (3) for our version of the Farkas' Lemma.

Moreover, in Sect. 2, we take a closer look at the reason for the lack of positive results. Farkas' Lemma may be viewed as an application of the Hahn-Banach Theorem (unavailable to Farkas in 1902): indeed, given linear forms $f_1, \ldots, f_k$, their positive cone

$$\mathcal{F} = \{a_1 f_1 + \cdots + a_k f_k : a_i \geq 0\}$$

is a convex set, so if $g \notin \mathcal{F}$ then it can be separated from $\mathcal{F}$ by a linear functional on $\mathbb{R}^{n*}$. But all linear functionals on $\mathbb{R}^{n*}$ are given by "evaluation maps". Thus there is a point $x \in \mathbb{R}^n$ such that

$$g(x) < 0 \leq f(x) \text{ for all } f \in \mathcal{F},$$

and in particular, $f_1(x) \geq 0, \ldots, f_k(x) \geq 0$ and $g(x) < 0$. This is the crucial point for the lack of Farkas type results in the multilinear (and other) settings: few linear functionals are "evaluation maps". We study evaluation maps on spaces of real-valued bilinear forms on $\mathbb{R}^n$ $(n \geq 2)$ (see Propositions 3 and 4), and this enables us to produce new examples (see Examples 1 and 2) of bilinear forms A_1, A_2 and B such that

$$A_1(x, y) \geq 0 \text{ and } A_2(x, y) \geq 0 \text{ imply } B(x, y) \geq 0, \text{ but } B \neq a_1 A_1 + a_2 A_2,$$

with non-negative a_i's.

1 Farkas' Lemma for simultaneously diagonalizable bilinear forms

In this section, $A_1, \ldots, A_k$, and B will be bilinear forms on $\mathbb{R}^n \times \mathbb{R}^n$. We will use the notation $B_x(y) = B(x, y)$ (and similarly for the bilinear forms A_i). Note that a bilinear form $A(x, y)$ can be written as $x^t[A]y$, where $[A] \in \mathbb{R}^{n \times n}$ is the coefficient matrix of A in the canonical basis: $[A]_{ij} = A(e_i, e_j)$. We shall prove Farkas' Lemma in the following form:

Theorem 1 *For $j = 1, ..., k$, let $A_j : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be simultaneously diagonalizable bilinear forms. Then*

$$B = a_1 A_1 + \cdots + a_k A_k, \text{ with all } a_i \geq 0$$

if and only if

$$A_1(x, y) \geq 0, \ldots, A_k(x, y) \geq 0 \text{ imply } B(x, y) \geq 0. \qquad (*)$$

Proof Clearly $\Rightarrow$) is trivial, so we must see $\Leftarrow$).

Note that for any $x \in \mathbb{R}^n$, we have

$$A_{1x}(y) \geq 0, \ldots, A_{kx}(y) \geq 0 \text{ imply } B_x(y) \geq 0,$$

so that by the linear Farkas' Lemma there are $a_1(x), \ldots, a_k(x) \geq 0$ such that

$$B_x = a_1(x) A_{1x} + \cdots + a_k(x) A_{kx}. \qquad (**)$$

The A_i's are diagonalized simultaneously by a matrix U whose columns form a basis of eigenvectors $\{u_1, \ldots, u_n\}$. Set

$$U = \begin{pmatrix} & \vdots & & \vdots & \\ u_1 & \vdots & \cdots & \vdots & u_n \\ & \vdots & & \vdots & \end{pmatrix},$$

and

$$U^{-1} = \begin{pmatrix} \ldots & v_1 & \ldots \\ \ldots & \vdots & \ldots \\ \ldots & v_n & \ldots \end{pmatrix}.$$

We have $\langle v_i, u_j \rangle = \delta_{ij}$, and for each $j = 1, \ldots, k$,

$$U^{-1}[A_j]U = \begin{pmatrix} A_j(v_1, u_1) & 0 & \ldots & 0 \\ 0 & A_j(v_2, u_2) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_j(v_n, u_n) \end{pmatrix}.$$

Note that if $r \neq s$, then for all j, $A_j(v_r, u_s) = 0$. Therefore from $(*)$ we deduce that $B(v_r, u_s) = 0$ for $r \neq s$ as well. Thus the same basis diagonalizes B. We have

$$U^{-1}[B]U = \begin{pmatrix} B(v_1, u_1) & 0 & \ldots & 0 \\ 0 & B(v_2, u_2) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & B(v_n, u_n) \end{pmatrix}.$$

Consider $1 = (1, \ldots, 1)$, and $z = (z_1, \ldots, z_n)$. Then for each $j = 1, \ldots, k$,

$$\begin{aligned} A_j((U^{-1})^t 1, Uz) &= 1^t U^{-1}[A_j]Uz \\ &= 1^t \begin{pmatrix} A_j(v_1, u_1) & 0 & \ldots & 0 \\ 0 & A_j(v_2, u_2) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_j(v_n, u_n) \end{pmatrix} z \\ &= A_j(v_1, u_1)z_1 + \cdots + A_j(v_n, u_n)z_n, \end{aligned}$$

and similarly, $B((U^{-1})^t 1, Uz) = B(v_1, u_1)z_1 + \cdots + B(v_n, u_n)z_n$. Now let $a_1 = a_1((U^{-1})^t 1) \geq 0, \ldots, a_k = a_k((U^{-1})^t 1) \geq 0$. Since, by $(**)$,

$$B_{(U^{-1})^t 1} = a_1 A_{1\,(U^{-1})^t 1} + \cdots + a_k A_{k\,(U^{-1})^t 1},$$

we have, for any $z \in \mathbb{R}^n$,

$$B((U^{-1})^t 1, Uz) = a_1 A_1((U^{-1})^t 1, Uz) + \cdots + a_k A_k((U^{-1})^t 1, Uz).$$

Now, for $r = 1, \ldots, n$, set $z = e_r$. We have

$$\begin{aligned} B(v_r, u_r) &= B((U^{-1})^t 1, Ue_r) \\ &= a_1 A_1((U^{-1})^t 1, Ue_r) + \cdots + a_k A_k((U^{-1})^t 1, Ue_r) \\ &= a_1 A_1(v_r, u_r) + \cdots + a_k A_k(v_r, u_r). \end{aligned}$$

Thus

$$U^{-1}[B]U = \begin{pmatrix} B(v_1,u_1) & 0 & \dots & 0 \\ 0 & B(v_2,u_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B(v_n,u_n) \end{pmatrix}$$
$$= a_1 \begin{pmatrix} A_1(v_1,u_1) & 0 & \dots & 0 \\ 0 & A_1(v_2,u_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_1(v_n,u_n) \end{pmatrix} + \cdots$$
$$\dots + a_k \begin{pmatrix} A_k(v_1,u_1) & 0 & \dots & 0 \\ 0 & A_k(v_2,u_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_k(v_n,u_n) \end{pmatrix}$$
$$= a_1U^{-1}[A_1]U + \cdots + a_kU^{-1}[A_k]U$$
$$= U^{-1}[a_1A_1 + \cdots + a_kA_k]U.$$

Therefore

$$B = a_1A_1 + \cdots + a_kA_k, \text{ with all } a_i \geq 0,$$

and the proof is complete. □

Note that by relaxing condition $(*)$ to

$$A_1(x, y) = 0, \dots, A_k(x, y) = 0 \text{ imply } B(x, y) = 0,$$

with the same proof but using (1) instead of the linear Farkas' Lemma, we have the following.

Theorem 2 *For $j = 1, ..., k$, let $A_j : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be simultaneously diagonalizable bilinear forms. Then*

$$B = a_1A_1 + \cdots + a_kA_k,$$

if and only if

$$A_1(x, y) = 0, \dots, A_k(x, y) = 0 \textit{ imply } B(x, y) = 0.$$

Also, since any number of symmetric commuting matrices are simultaneously diagonalizable [8, Theorem 1.3.21], we have the following corollary.

Corollary 1 *If $A_1, \dots, A_k$, are symmetric bilinear forms on $\mathbb{R}^n \times \mathbb{R}^n$ defined by commuting matrices, then*

$$B = a_1A_1 + \cdots + a_kA_k, \textit{ with all } a_i \geq 0$$

if and only if

$$A_1(x, y) \geq 0, \dots, A_k(x, y) \geq 0 \textit{ imply } B(x, y) \geq 0.$$

Remark 1 We note that an analogous result can be obtained in the setting of a Hilbert space H as follows:

If $A_1, \ldots, A_k$, are symmetric bilinear forms on $H \times H$ defined by $A_i(x, y) = \langle y, T_i x \rangle$ where, for $i = 1, \ldots, k$, the T_i's are compact self-adjoint commuting operators, then

$$B = a_1 A_1 + \cdots + a_k A_k, \text{ with all } a_i \geq 0$$

if and only if

$$A_1(x, y) \geq 0, \ldots, A_k(x, y) \geq 0 \text{ imply } B(x, y) \geq 0.$$

2 Evaluation maps over spaces of bilinear forms

As we have mentioned above, linear forms on $\mathbb{R}^{n*}$ identify with elements of $\mathbb{R}^n$, and thus are all evaluation maps: $\gamma \mapsto \gamma(x)$. This is far from the case in most function spaces. For example, continuous linear functionals on $C[0, 1]$ are identified with regular Borel measures, while the evaluation maps are the deltas: $\delta_x(f) = f(x)$. The two convex sets

$$A = \{f \in C[0, 1] : f \text{ is strictly increasing }\}$$
$$\text{and } B = \{g \in C[0, 1] : g \text{ is strictly decreasing }\}$$

can be separated by the measure $\delta_x - \delta_y$ for any $0 \leq x \neq y \leq 1$, although they cannot be separated by any evaluation map δ_x.

We now characterize the evaluation maps on the space of bilinear forms $\mathcal{L}^2(\mathbb{R}^n)$ and on the space of *symmetric* bilinear forms $\mathcal{L}_s^2(\mathbb{R}^n)$ in order to construct examples of positive cones

$$\mathcal{F} = \{a_1 A_1 + \cdots + a_n A_n : a_i \geq 0\}$$

and $B \notin \mathcal{F}$ which cannot be separated by evaluation maps.

We consider $\mathcal{L}^2(\mathbb{R}^n)$ as $\mathbb{R}^{n \times n}$, the space of $n \times n$ matrices with inner product:

$$\langle A, B \rangle = \sum_i \sum_j a_{ij} b_{ij} = tr(AB^t),$$

where $A = (a_{ij})$ and $B = (b_{ij})$. For a matrix A, we denote by $tr(A)$ the trace of A and by $rk(A)$ the rank of A. Any $\varphi \in \mathcal{L}^2(\mathbb{R}^n)^*$ can be represented as $\varphi(A) = tr(AN^t)$ for some $n \times n$ matrix N. As we will see below, in the symmetric setting, any $\varphi \in \mathcal{L}_s^2(\mathbb{R}^n)^*$ can be represented as $\varphi(A) = tr(AS)$, with a symmetric matrix S. We say that φ is an *evaluation map* if there are x and y in $\mathbb{R}^n$ such that $\varphi(A) = A(x, y) = x^t Ay$.

We characterize the evaluation maps in $\mathcal{L}^2(\mathbb{R}^n)^*$ and $\mathcal{L}_s^2(\mathbb{R}^n)^*$ as matrices N and S as follows.

Proposition 3 *$\varphi \in \mathcal{L}^2(\mathbb{R}^n)^*$ is an evaluation map if and only if $\varphi(A) = tr(AN^t)$, where N has rank less than or equal to one.*

Proof $x^t Ay = \sum_i \sum_j a_{ij} x_i y_j$, so set $n_{ij}^t = x_i y_j$ for the entries of N^t. Thus

$$rk(N) = rk \begin{pmatrix} x_1 y_1 & \ldots & x_1 y_n \\ x_2 y_1 & \ldots & x_2 y_n \\ \vdots & \ddots & \vdots \\ x_n y_1 & \ldots & x_n y_n \end{pmatrix} \leq rk \begin{pmatrix} y_1 & y_2 & \ldots & y_n \\ y_1 & y_2 & \ldots & y_n \\ \vdots & \vdots & \ddots & \vdots \\ y_1 & y_2 & \ldots & y_n \end{pmatrix} \leq 1.$$

N has rank 0 if $\varphi = 0$ and rank one otherwise.

Conversely, if N has rank one, we can write

$$N^t = \begin{pmatrix} x_1 y_1 & \dots & x_1 y_n \\ x_2 y_1 & \dots & x_2 y_n \\ \vdots & \ddots & \vdots \\ x_n y_1 & \dots & x_n y_n \end{pmatrix},$$

for some $y \in \mathbb{R}^n$, and so $n^t_{ij} = x_i y_j$, and N produces an evaluation map. □

We now discuss the symmetric setting. We note that we can associate $\mathcal{L}^2_s(\mathbb{R}^n)$ with the space $\mathbb{R}^{n\times n}_s$ of symmetric $n \times n$ matrices. Also, any continuous linear functional $\varphi : \mathcal{L}^2_s(\mathbb{R}^n) \to \mathbb{R}$ is given by $\varphi(M) = tr(M[\varphi]^t)$ for a suitable symmetric $n \times n$ matrix $[\varphi]$. Indeed, since M is symmetric, if $\varphi(M) = tr(MN^t)$, we can replace N by a symmetric matrix as follows.

$$\begin{aligned} \varphi(M) = tr(MN^t) &= \frac{tr(MN^t) + tr((MN^t)^t)}{2} \\ &= \frac{tr(MN^t) + tr(NM)}{2} \\ &= \frac{tr(MN^t) + tr(MN)}{2} \\ &= tr\left(M \frac{N^t + N}{2}\right) \\ &= tr(M[\varphi]), \text{ where } [\varphi] = \frac{N^t + N}{2}. \end{aligned}$$

For each $x, y \in \mathbb{R}^n$, we will identify an *evaluation functional* $E(x, y)$ on $\mathbb{R}^{n\times n}_s$ by

$$M \in \mathbb{R}^{n\times n}_s \rightsquigarrow E(x, y)(M) \equiv x^t M y.$$

We can associate $E(x, y)$ to the symmetric $n \times n$ matrix $[E(x, y)] \in \mathbb{R}^{n\times n}_s$ where

$$[E(x, y)]_{i,j} = \frac{x_i y_j + x_j y_i}{2}.$$

Observe that if a matrix $N \in \mathbb{R}^{n\times n}_s$ represents an evaluation map, then so does the matrix UNU^t, where $U \in \mathcal{O}(n)$ (the orthogonal group in $\mathbb{R}^{n\times n}$). To see this, if $N = E(x, y)$ is an evaluation map, so that $E(x, y)(M) = tr(MN^t)$, then for any such U,

$$\begin{aligned} tr(M(UNU^t)^t) &= tr(MUN^tU^t) = tr(MUNU^t) \\ &= tr(U^t MUN) = E(x, y)(U^t MU) \\ &= x^t U^t MU y = (Ux)^t M(Uy) = E(Ux, Uy)(M), \end{aligned}$$

where the second equality holds since N is symmetric.

Therefore, if N is a symmetric $n \times n$ matrix, we can find an orthogonal matrix U so that UNU^t has a diagonal representation. In particular, for any such diagonal matrix $[E(x, y)] =$

$$\begin{bmatrix} x_1 y_1 & 0 & \cdots & 0 \\ 0 & x_2 y_2 & \cdots & 0 \\ 0 & \cdots & x_i y_i & \cdots \\ 0 & 0 & \cdots & x_n y_n \end{bmatrix}$$

In fact, at most two of the diagonal entries must be non-zero. Indeed, suppose that there are three non-zero entries, $\lambda_i = x_i y_i$, $\lambda_j = x_j y_j$, and $\lambda_k = x_k y_k$. Since the off-diagonal entries are all 0, it follows that

$$\lambda_j \frac{x_i}{x_j} + \lambda_i \frac{x_j}{x_i} = 0,$$
$$\lambda_k \frac{x_i}{x_k} + \lambda_i \frac{x_k}{x_i} = 0,$$
$$\text{and } \lambda_k \frac{x_j}{x_k} + \lambda_j \frac{x_k}{x_j} = 0.$$

As a consequence, $\lambda_j x_i^2 + \lambda_i x_j^2 = 0$, $\lambda_k x_i^2 + \lambda_i x_k^2 = 0$, and $\lambda_k x_j^2 + \lambda_j x_k^2 = 0$. Since none of the entries is 0, it follows that all three of λ_i, λ_j, and λ_k are of opposite sign, a clear impossibility. In addition, the argument shows that if there are just two non-zero diagonal entries, say with $i = 1$ and $j = 2$, then $\lambda_1 \lambda_2 < 0$.

Consequently, if $\lambda_1 = \lambda_2 = 0$, then the linear form $\varphi = E(\vec{0}, \vec{0})$, and if only $\lambda_1 \neq 0$, then $\varphi = E(\lambda_1 e_1, e_1)$. The third and last case occurs if $\lambda_1 \neq 0 \neq \lambda_2$. If say $\lambda_1 > 0 > \lambda_2$, then by using the facts that λ_2 is negative and that we are dealing with symmetric matrices it follows that $\varphi = E(\sqrt{\frac{-\lambda_1}{\lambda_2}}\, e_1 + e_2, \sqrt{-\lambda_1 \lambda_2}\, e_1 + \lambda_2 e_2)$.

Thus, given an evaluation map $E(x, y)$, we have now seen that to the associated matrix $[E(x, y)] \in \mathbb{R}_s^{n\times n}$ there is an orthogonal matrix U such that $U[E(x, y)]U^t$ is diagonal which also represents an evaluation map. Summarizing, we have the following.

Proposition 4 *The square matrix $S \in \mathbb{R}_s^{n\times n}$ represents an evaluation map if and only if S is one of the following:*

(i) $S = 0$, *or*
(ii) *S has only one non-zero eigenvalue, or*
(iii) *S has exactly two non-zero eigenvalues which are of opposite sign.*

We now construct counterexamples to Farkas' Lemma for both cases, the four dimensional $\mathcal{L}^2(\mathbb{R}^2)$ and the three dimensional $\mathcal{L}_s^2(\mathbb{R}^2)$. Note that according to our characterizations, evaluation functionals over $\mathcal{L}^2(\mathbb{R}^2)$ are of the form $\varphi(A) = \langle A, N\rangle$ where $\det N = 0$, while evaluation functionals over $\mathcal{L}_s^2(\mathbb{R}^2)$ are of the form $\varphi(A) = \langle A, S\rangle$ where $\det S \leq 0$. We begin with counterexamples in $\mathcal{L}^2(\mathbb{R}^2)$.

Example 1 Let $[I]^\perp \subset \mathcal{L}^2(\mathbb{R}^2)$ be the orthogonal complement of the line spanned by the identity matrix, and $\{A_1, A_2, A_3\}$ an orthonormal basis of $[I]^\perp$. We consider $B = A_1 + A_2 + A_3 - \frac{\varepsilon}{\sqrt{2}} I$ where $\varepsilon > 0$ will be chosen later. Clearly $B \notin \mathcal{F} \equiv \{a_1 A_1 + a_2 A_2 + a_3 A_3 : a_i \geq 0\}$. Let φ be a linear form separating B from $\mathcal{F}$ chosen so that

$$\varphi(B) < 0 \leq \varphi(X) \text{ for all } X \in \mathcal{F}.$$

φ is given by the matrix N: $\varphi(X) = \langle X, N\rangle$. We may suppose N has norm one. We will show that if $\varepsilon > 0$ is chosen to be sufficiently small, then φ cannot be an evaluation map. We have

$$0 > \varphi(B) = \langle B, N\rangle = \langle A_1 + A_2 + A_3, N\rangle - \frac{\varepsilon}{\sqrt{2}} \langle I, N\rangle,$$

so

$$\begin{aligned} \langle A_1, N\rangle + \langle A_2, N\rangle + \langle A_3, N\rangle &= \langle A_1 + A_2 + A_3, N\rangle \\ &< \frac{\varepsilon}{\sqrt{2}} \langle I, N\rangle \quad \leq \varepsilon. \end{aligned}$$

Thus

$$\begin{aligned}\langle A_1, N\rangle^2 + \langle A_2, N\rangle^2 + \langle A_3, N\rangle^2 &\leq (\langle A_1, N\rangle + \langle A_2, N\rangle + \langle A_3, N\rangle)^2 \\ &< \varepsilon^2.\end{aligned}$$

The first inequality follows from the fact that the right part is equal to the left part plus the double products of $\langle A_i, N\rangle\langle A_j, N\rangle$, which are terms greater than or equal to zero since $0 \leq \varphi(X)$ for all $X \in \mathcal{F}$.

Also,

$$1 = \|N\|^2 = \langle A_1, N\rangle^2 + \langle A_2, N\rangle^2 + \langle A_3, N\rangle^2 + \left\langle \frac{I}{\sqrt{2}}, N\right\rangle^2,$$

so

$$\left\langle \frac{I}{\sqrt{2}}, N\right\rangle^2 = 1 - \left(\langle A_1, N\rangle^2 + \langle A_2, N\rangle^2 + \langle A_3, N\rangle^2\right) > 1 - \varepsilon^2.$$

Thus $\sqrt{2}\sqrt{1-\varepsilon^2} < \langle I, N\rangle$. Now by the parallelogram law we have

$$\begin{aligned}\|I - N\|^2 &= 2\left(\|I\|^2 + \|N\|^2\right) - \|I + N\|^2 \\ &= 6 - \left(\|I\|^2 + 2\langle I, N\rangle + \|N\|^2\right) \\ &= 3 - 2\langle I, N\rangle \\ &< 3 - 2\sqrt{2}\sqrt{1-\varepsilon^2},\end{aligned}$$

which can be made smaller than one for small ε because $3 - 2\sqrt{2} < 0.18$. Thus, for such ε, $\|I - N\|$ is smaller than one, and N is invertible. By Proposition 3, φ cannot be an evaluation map.

For the symmetric case we have the following.

Example 2 Let $[I]^\perp \subset \mathcal{L}_s^2(\mathbb{R}^2)$ be the orthogonal complement of the line spanned by the identity matrix, and $\{A_1, A_2\}$ be an orthonormal basis of $[I]^\perp$. We consider $B = A_1 + A_2 - \frac{\varepsilon}{\sqrt{2}} I$ where $\varepsilon > 0$ will be chosen later. Clearly B is not in $\mathcal{F} \equiv \{a_1 A_1 + a_2 A_2 : a_i \geq 0\}$. Let φ be such that

$$\varphi(B) < 0 \leq \varphi(X) \text{ for all } X \in \mathcal{F}.$$

φ is given by the symmetric matrix S, which we may suppose has norm one: $\varphi(X) = \langle X, S\rangle$. Picking a suitable small enough ε as we have done in the previous example, we have $\|I - S\| < 1$. But then the determinant of S must be positive: if $\det S \leq 0$, the line segment joining I and S would, by the intermediate value theorem, contain a matrix X with $\det X = 0$, at a distance smaller than one from the identity. By Proposition 4, φ cannot be an evaluation map.

Note that no basis of $[I]^\perp$ can be diagonalized simultaneously. Otherwise, all matrices would be simultaneously diagonalizable.

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