# Optimal ratcheting of dividends in a Brownian risk model 

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#### Abstract

We study the problem of optimal dividend payout from a surplus process governed by Brownian motion with drift under the additional constraint of ratcheting, i.e. the dividend rate can never decrease. We solve the resulting two-dimensional optimal control problem, identifying the value function to be the unique viscosity solution of the corresponding Hamilton-Jacobi-Bellman equation. For finitely many admissible dividend rates we prove that threshold strategies are optimal, and for any finite continuum of admissible dividend rates we establish the $\varepsilon$-optimality of curve strategies. This work is a counterpart of [2], where the ratcheting problem was studied for a compound Poisson surplus process with drift. In the present Brownian setup, calculus of variation techniques allow to obtain a much more explicit analysis and description of the optimal dividend strategies. We also give some numerical illustrations of the optimality results.


## 1 Introduction

The identification of the optimal way to pay out dividends from a surplus process to shareholders is a classical topic in actuarial science and mathematical finance. There is a natural trade-off between paying out gains as dividends to shareholders early and at the same time leaving sufficient surplus in order to safeguard future adverse developments and avoid ruin. Depending on risk preferences, the concrete situation and the simultaneous exposure to other risk factors such a problem can be formally stated in various different ways in terms of objective functions and constraints. In this paper we would like to follow the actuarial tradition of considering the surplus process as the free capital in an insurance portfolio at any point in time, and the goal is to maximize the expected sum of discounted dividend payments that can be paid until the surplus process goes below 0 (which is called the time of ruin). In such a formulation, the problem goes back to de Finetti [15] and Gerber [18], and has since then been studied in many variants concerning the nature of the underlying surplus process and constraints on the type of admissible dividend payment strategies, see e.g. Albrecher \& Thonhauser [4] and Avanzi [8] for an overview. From a mathematical perspective, the problem turns out to be quite challenging, and was cast into the framework of modern stochastic control theory and the concept of viscosity solutions for corresponding Hamilton-Jacobi-Bellman equations over the last years, cf. Schmidli (25) and Azcue \& Muler (10].

Among the variants of the general problem is to look for the optimal dividend payment strategy if the rate at which dividends are paid can never be reduced. This ratcheting constraint has often been brought up by practitioners and is in part motivated by the psychological

[^0]effect that shareholders are likely to be unhappy about a reduction of dividend payments over time (see e.g. Avanzi et al. 9 for a discussion). One crucial question in this context is how much of the expected discounted dividends until ruin is lost if one respects such a ratcheting constraint, if that ratcheting is done in an optimal way. A first step in that direction was done in Albrecher et al. 3, where the consequences of ratcheting were studied under the simplifying assumption that a dividend rate can be fixed in the beginning and can be augmented only once during the lifetime of the process (concretely, when the surplus process hits some optimally chosen barrier for the first time). The analysis in that paper was both for a Brownian surplus process as well as for a surplus process of compound Poisson type. In our recent paper [2], we then provided the analysis and solution for the general ratcheting problem for the latter compound Poisson process, and it turned out that the optimal ratcheting dividend strategy does not lose much efficiency compared to the unconstrained optimal dividend payout performance, and also that the one-step ratcheting strategy studied earlier compares remarkably well to the overall optimal ratcheting solution. In this paper we would like to address the general ratcheting problem for the Brownian risk model. Such a model can be seen as a diffusion approximation of the compound Poisson risk model, but is also interesting in its own right. In particular, the fact that ruin with zero initial capital is immediate often leads to a more amenable analysis of stochastic control problems. In addition, the convergence of optimal strategies from a compound Poisson setting to the one for the diffusion approximation can be quite delicate, see e.g. Bäuerle [12], see also Cohen and Young [13] for a recent convergence rate analysis of simple uncontrolled ruin probabilities towards its counterparts for the diffusion limit.

On the mathematical level, the general ratcheting formulation leads to a fully two-dimensional stochastic control problem with all its related challenges, and it is only recently that in the context of insurance risk theory some first two-dimensional problems became amenable for analysis, see e.g. Albrecher et al. [1], Gu et al. [22], Grandits [21] and Azcue et al. [11]. In the present contribution we would like to exploit the more amenable nature of the ratcheting problem in the diffusion setting that push the analysis considerably further than was possible in [2]. In particular, we will use calculus of variation techniques to identify quite explicit formulas for the candidates of optimal strategies and provide various optimality results.

We would like to mention that optimal ratcheting strategies have been investigated in the framework of lifetime consumption in the mathematical finance literature, see e.g. Dybvig 16, Elie and Touzi [17, Jeon et al. [23] and more recently Angoshtari et al. [5]. However, the concrete model setup and correspondingly also the involved techniques are quite different to the one of the present paper.

The remainder of the paper is organized as follows. Section 2 introduces the model and the detailed formulation of the problem. It also provides some first basic results on properties of the value function under consideration. Section 3 derives the Hamilton-Jacobi-Bellman equations and characterization theorems for the value function for both a closed interval as well as a finite discrete set of admissible dividend payment rates. In Section 4 we prove that the optimal value function of the problem for discrete sets convergences to the one for a continuum of admissible dividend rates as the mesh size of the finite set tends to zero. In Section 5 we show that for finitely many admissible dividend rates, there exists an optimal strategy for which the change and non-change regions have only one connected component (this corresponds to the extension of one-dimensional threshold strategies to the two-dimensional case). We also provide an implicit equation defining the optimal threshold function for this case. Subsequently, we turn to the case of a continuum of admissible dividend rates and use calculus of variation techniques to identify the optimal curve splitting the state space into a change and a non-change region as the unique solution of an ordinary differential equation. We show that the corresponding dividend strategy is $\varepsilon$-optimal, in the sense that there exists a known sequence of curves such that the corresponding value functions converge uniformly to the optimal value function of the problem.

Section 6 contains a numerical illustration of the optimal strategy and its performance relative to the one of for the unconstrained dividend problem and the one where the dividend rate can only be increased once. Section 7 concludes.

Some technical proofs are delegated to an appendix.

## 2 Model and basic results

Assume that the surplus process of a company is given by a Brownian motion with drift

$$
X_{t}=x+\mu t+\sigma W_{t}
$$

where $W_{t}$ is a standard Brownian motion, and $\mu, \sigma>0$ are given constants. Let $\left(\Omega, \mathcal{F},\left(\mathcal{F}_{t}\right)_{t \geq 0}, \mathcal{P}\right)$ be the complete probability space generated by the process $X_{t}$.

The company uses part of the surplus to pay dividends to the shareholders with rates in a set $S \subset[0, \bar{c}]$, where $0 \leq \bar{c} \in S$ is the maximum dividend rate possible. Let us denote by $C_{t}$ the rate at which the company pays dividends at time $t$. Given an initial surplus $X_{0}=x$ and a minimum dividend rate $c \in S$ at $t=0$, a dividend ratcheting strategy is given by $C=\left(C_{t}\right)_{t>0}$ and it is admissible if it is non-decreasing, right-continuous, adapted with respect to the filtration $\left(\mathcal{F}_{t}\right)_{t \geq 0}$ and it satisfies $C_{t} \in S$ for all $t$. The controlled surplus process can be written as

$$
\begin{equation*}
X_{t}^{C}=X_{t}-\int_{0}^{t} C_{s} d s \tag{1}
\end{equation*}
$$

Define $\Pi_{x, c}^{S}$ as the set of all admissible dividend ratcheting strategies with initial surplus $x \geq 0$ and minimum initial dividend rate $c \in S$. Given $C \in \Pi_{x, c}^{S}$, the value function of this strategy is given by

$$
J(x ; C)=\mathbb{E}\left[\int_{0}^{\tau} e^{-q s} C_{s} d s\right],
$$

where $q>0$ and $\tau=\inf \left\{t \geq 0: X_{t}^{C}<0\right\}$ is the ruin time. Hence, for any initial surplus $x \geq 0$ and initial dividend rate $c$, our aim is to maximize

$$
\begin{equation*}
V^{S}(x, c)=\sup _{C \in \Pi_{x, c}^{S}} J(x ; C) . \tag{2}
\end{equation*}
$$

It is immediate to see that $V^{S}(0, c)=0$ for all $c \in S$.
Remark 2.1 The dividend optimization problem without the ratcheting constraint, that is where the dividend strategy $C=\left(C_{t}\right)_{t \geq 0}$ is not necessarily non-decreasing, was studied intensively in the literature (see e.g. Gerber and Shiu [19]). Unlike the ratcheting optimization problem, this non-ratcheting problem is one-dimensional. If $V_{N R}(x)$ denotes its optimal value function, then clearly $V^{S}(x, c) \leq V_{N R}(x)$ for all $x \geq 0$ and $c \in S \subset[0, \bar{c}]$. The function $V_{N R}$ is increasing, concave, twice continuously differentiable with $V_{N R}(0)=0$ and $\lim _{x \rightarrow \infty} V_{N R}(x)=\bar{c} / q$; so it is Lipschitz with Lipschitz constant $V_{N R}^{\prime}(0)$.

The following Lemma states the dynamic programming principle, its proof is similar to the one of Lemma 1.2 in Azcue and Muler [10.

Lemma 2.1 Given any stopping time $\widetilde{\tau}$, we can write

$$
V^{S}(x, c)=\sup _{C \in \Pi_{\tilde{x}, c}^{S}} \mathbb{E}\left[\int_{0}^{\tau \wedge \tilde{\tau}} e^{-q s} C_{s} d s+e^{-q(\tau \wedge \widetilde{\tau})} V^{S}\left(X_{\tau \wedge \widetilde{\tau}}^{C}, C_{\tau \wedge \widetilde{\tau}}\right)\right]
$$

We now state a straightforward result regarding the boundedness and monotonicity of the optimal value function.

Proposition 2.2 The optimal value function $V^{S}(x, c)$ is bounded by $\bar{c} / q$, non-decreasing in $x$ and non-increasing in $c$.

Proof. Since the discounted value of paying the maximum rate $\bar{c}$ up to infinity is $\bar{c} / q$, we conclude the boundedness result.

On the one hand $V^{S}(x, c)$ is non-increasing in $c$ because given $c_{1}<c_{2}$ we have $\Pi_{x, c_{2}}^{S} \subset \Pi_{x, c_{1}}^{S}$ for any $x \geq 0$. On the other hand, given $x_{1}<x_{2}$ and an admissible ratcheting strategy $C^{1} \in \Pi_{x_{1}, c}^{S}$ for any $c \in S$, let us define $C^{2} \in \Pi_{x_{2}, c}^{S}$ as $C_{t}^{2}=C_{t}^{1}$ until the ruin time of the controlled process $X_{t}^{C^{1}}$ with $X_{0}^{C^{1}}=x_{1}$, and pay the maximum rate $\bar{c}$ afterwards. Thus, $J\left(x ; C_{1}\right) \leq J\left(x ; C_{2}\right)$ and we have the result.

The Lipschitz property of the function $V_{N R}$ introduced in Remark 2.1 can now be used to prove a first result on the regularity of the function $V^{S}$.

Proposition 2.3 There exists a constant $K>0$ such that

$$
0 \leq V^{S}\left(x_{2}, c_{1}\right)-V^{S}\left(x_{1}, c_{2}\right) \leq K\left[\left(x_{2}-x_{1}\right)+\left(c_{2}-c_{1}\right)\right]
$$

for all $0 \leq x_{1} \leq x_{2}$ and $c_{1}, c_{2} \in S$ with $c_{1} \leq c_{2}$.
The proof is given in the appendix.

## 3 Hamilton-Jacobi-Bellman equations

In this section we introduce the Hamilton-Jacobi-Bellman (HJB) equation of the ratcheting problem for $S \subset[0, \infty)$, when $S$ is either a closed interval or a finite set. We show that the optimal value function $V$ defined in (2) is the unique viscosity solution of the corresponding HJB equation with boundary condition $\bar{c} / q$ when $x$ goes to infinity, where $\bar{c}=\max S$.

First, consider the case $S=\{c\}$. In this case, the unique admissible strategy consists of paying a constant dividend rate $c$ up to the ruin time. Correspondingly, the value function $V^{\{c\}}(x, c)$ is the unique solution of the second order differential equation

$$
\begin{equation*}
\mathcal{L}^{c}(W):=\frac{\sigma^{2}}{2} \partial_{x x} W+(\mu-c) \partial_{x} W-q W+c=0 \tag{3}
\end{equation*}
$$

with boundary conditions $V^{\{c\}}(0, c)=0$ and $\lim _{x \rightarrow \infty} V^{\{c\}}(x, c)=c / q$. The solutions of (3) are of the form

$$
\begin{equation*}
\frac{c}{q}+a_{1} e^{\theta_{1}(c) x}+a_{2} e^{\theta_{2}(c) x} \text { with } a_{1}, a_{2} \in \mathbb{R} \tag{4}
\end{equation*}
$$

where $\theta_{1}(c)>0$ and $\theta_{2}(c)<0$ are the roots of the characteristic equation

$$
\frac{\sigma^{2}}{2} z^{2}+(\mu-c) z-q=0
$$

associated to the operator $\mathcal{L}^{c}$, that is

$$
\begin{equation*}
\theta_{1}(c):=\frac{c-\mu+\sqrt{(c-\mu)^{2}+2 q \sigma^{2}}}{\sigma^{2}}, \quad \theta_{2}(c):=\frac{c-\mu-\sqrt{(c-\mu)^{2}+2 q \sigma^{2}}}{\sigma^{2}} . \tag{5}
\end{equation*}
$$

In the following remark, we state some basic properties of $\theta_{1}$ and $\theta_{2}$.
Remark 3.1 We have that

1. $\theta_{1}(c)=-\theta_{2}(c)$ if $c=\mu$ and $\theta_{1}^{2}(c) \geq \theta_{2}^{2}(c)$ if, and only if, $c-\mu \geq 0$.
2. $\theta_{1}^{\prime}(c)=\frac{1}{\sigma^{2}}\left(1+\frac{c-\mu}{\sqrt{(c-\mu)^{2}+2 q \sigma^{2}}}\right)$ and $\theta_{2}^{\prime}(c)=\frac{1}{\sigma^{2}}\left(1-\frac{c-\mu}{\sqrt{(c-\mu)^{2}+2 q \sigma^{2}}}\right)$, so $\theta_{1}^{\prime}(c), \theta_{2}^{\prime}(c) \in\left(0, \frac{2}{\sigma^{2}}\right)$.

The solutions of $\mathcal{L}^{c}(W)=0$ with boundary condition $W(0)=0$ are of the form

$$
\begin{equation*}
\frac{c}{q}\left(1-e^{\theta_{2}(c) x}\right)+a\left(e^{\theta_{1}(c) x}-e^{\theta_{2}(c) x}\right) \text { with } a \in \mathbb{R} \tag{6}
\end{equation*}
$$

And finally, the unique solution of $\mathcal{L}^{c}(W)=0$ with boundary conditions $W(0)=0$ and $\lim _{x \rightarrow \infty}$ $W(x)=c / q$ corresponds to $a=0$, so that

$$
\begin{equation*}
V^{\{c\}}(x, c)=\frac{c}{q}\left(1-e^{\theta_{2}(c) x}\right) . \tag{7}
\end{equation*}
$$

We have that $V^{\{c\}}(\cdot, c)$ is increasing and concave.
Remark 3.2 Given a set $S \subset[0, \infty)$ with $\bar{c}=\max S<\infty$, we have that

$$
V^{S}(x, c) \geq V^{\{\bar{c}\}}(x, \bar{c})=\frac{\bar{c}}{q}\left(1-e^{\theta_{2}(\bar{c}) x}\right)
$$

and so, by Remark 2.1, we conclude that $\lim _{x \rightarrow \infty} V^{S}(x, c)=\bar{c} / q$ for any $c \in S$.

### 3.1 Hamilton-Jacobi-Bellman equations for closed intervals

Let us now consider the case $S=[\underline{c}, \bar{c}]$ with $0 \leq \underline{c}<\bar{c}$. The HJB equation associated to (2) is given by

$$
\begin{equation*}
\max \left\{\mathcal{L}^{c}(u)(x, c), \partial_{c} u(x, c)\right\}=0 \text { for } x \geq 0 \text { and } \underline{c} \leq c \leq \bar{c} \tag{8}
\end{equation*}
$$

We say that a function $f:[0, \infty) \times[\underline{c}, \bar{c}) \rightarrow \mathbb{R}$ is $(2,1)$-differentiable if $f$ is continuously differentiable and $\partial_{x} f(\cdot, c)$ is continuously differentiable.

Definition 3.1 (a) A locally Lipschitz function $\bar{u}:[0, \infty) \times[\underline{c}, \bar{c}) \rightarrow \mathbb{R}$ where $0 \leq \underline{c}<\bar{c}$ is a viscosity supersolution of (8) at $(x, c) \in(0, \infty) \times[\underline{c}, \bar{c})$, if any (2,1)-differentiable function $\varphi:[0, \infty) \times[\underline{c}, \bar{c}) \rightarrow \mathbb{R}$ with $\varphi(x, c)=\bar{u}(x, c)$ such that $\bar{u}-\varphi$ reaches the minimum at $(x, c)$ satisfies

$$
\max \left\{\mathcal{L}^{c}(\varphi)(x, c), \partial_{c} \varphi(x, y)\right\} \leq 0
$$

The function $\varphi$ is called a test function for supersolution at $(x, c)$.
(b) A function $\underline{u}:[0, \infty) \times[\underline{c}, \bar{c}) \rightarrow \mathbb{R}$ is a viscosity subsolution of (8) at $(x, c) \in(0, \infty) \times[\underline{c}, \bar{c})$, if any (2,1)-differentiable function $\psi:[0, \infty) \times[\underline{c}, \bar{c}) \rightarrow \mathbb{R}$ with $\psi(x, c)=\underline{u}(x, c)$ such that $\underline{u}-\psi$ reaches the maximum at $(x, c)$ satisfies

$$
\max \left\{\mathcal{L}^{c}(\psi)(x, c), \partial_{c} \psi(x, c)\right\} \geq 0
$$

The function $\psi$ is called a test function for subsolution at $(x, c)$.
(c) A function $u:[0, \infty) \times[\underline{c}, \bar{c}) \rightarrow \mathbb{R}$ which is both a supersolution and subsolution at $(x, c) \in[0, \infty) \times[\underline{c}, \bar{c})$ is called a viscosity solution of (8) at $(x, c)$.
Remark 3.3 Note that, by $22, V^{[0, \bar{c}]}(x, c)=V^{[c, \bar{c}]}(x, c)$ for all $0 \leq \underline{c} \leq c \leq \bar{c}$, so in order to simplify the notation we define $V(x, c):=V^{[c, \bar{c}]}(x, c):[0, \infty) \times[\underline{c}, \bar{c}) \rightarrow \mathbb{R}$.

We first prove that $V$ is a viscosity solution of the corresponding HJB equation.
Proposition 3.1 $V$ is a viscosity solution of (8) in $(0, \infty) \times[\underline{c}, \bar{c})$.
The proof is given in the appendix.
Note that by definition of ratcheting $V(x, \bar{c})$ corresponds to the value function of the strategy that constantly pays dividends at rate $\bar{c}$, with initial surplus $x$. So, by (7),

$$
\begin{equation*}
V(x, \bar{c})=V^{\{\bar{c}\}}(x, \bar{c}) . \tag{9}
\end{equation*}
$$

Let us now state the comparison result for viscosity solutions.

Lemma 3.2 Assume that (i) $\underline{u}$ is a viscosity subsolution and $\bar{u}$ is a viscosity supersolution of the HJB equation (8) for all $x>0$ and for all $c \in[\underline{c}, \bar{c}$ ) with $0 \leq \underline{c}<\bar{c}$, (ii) $\underline{u}$ and $\bar{u}$ are non-decreasing in the variable $x$ and Lipschitz in $[0, \infty) \times[\underline{c}, \bar{c})$, and (iii) $\underline{u}(0, c)=\bar{u}(0, c)=0$, $\lim _{x \rightarrow \infty} \underline{u}(x, c) \leq \bar{c} / q \leq \lim _{x \rightarrow \infty} \bar{u}(x, c)$. Then $\underline{u} \leq \bar{u}$ in $[0, \infty) \times[\underline{c}, \bar{c})$.

The proof is given in the appendix.
The following characterization theorem is a direct consequence of the previous lemma, Remark 3.2 and Proposition 3.1.

Theorem 3.3 The optimal value function $V$ is the unique function non-decreasing in $x$ that is a viscosity solution of (8) in $(0, \infty) \times[\underline{c}, \bar{c})$ with $V(0, c)=0$ and $\lim _{x \rightarrow \infty} V(x, c)=\bar{c} / q$ for $c \in[\underline{c}, \bar{c})$.

From Definition 2, Lemma 3.2, and Remark 3.2 together with Proposition 3.1, we also get the following verification theorem.

Theorem 3.4 Consider $S=[\underline{c}, \bar{c}]$ and consider a family of strategies

$$
\left\{C_{x, c} \in \Pi_{x, c}^{S}:(x, c) \in[0, \infty) \times[\underline{c}, \bar{c}]\right\}
$$

If the function $W(x, c):=J\left(x ; C_{x, c}\right)$ is a viscosity supersolution of the HJB equation (8) in $(0, \infty) \times[\underline{c}, \bar{c})$ with $W(0, c)=0$ and $\lim _{x \rightarrow \infty} W(x, c)=\bar{c} / q$, then $W$ is the optimal value function V. Also, if for each $k \geq 1$ there exists a family of strategies $\left\{C_{x, c}^{k} \in \Pi_{x, c}^{S}:(x, c) \in[0, \infty) \times[\underline{c}, \bar{c}]\right\}$ such that $W(x, c):=\lim _{k \rightarrow \infty} J\left(x ; C_{x, c}^{k}\right)$ is a viscosity supersolution of the HJB equation (8) in $(0, \infty) \times[\underline{c}, \bar{c})$ with $W(0, c)=0$ and $\lim _{x \rightarrow \infty} W(x, c)=\bar{c} / q$, then $W$ is the optimal value function $V$.

### 3.2 Hamilton-Jacobi-Bellman equations for finite sets

Let us now consider the case

$$
S=\left\{c_{1}, c_{2}, \ldots, c_{n}\right\}
$$

where $0 \leq c_{1}<c_{2}<\ldots<c_{n}=\bar{c}$. Note that $V^{S}\left(x, c_{i}\right)=V^{\left\{c_{i}, c_{i+1}, \ldots, c_{n}\right\}}\left(x, c_{i}\right)$. We simplify the notation as follows:

$$
\begin{equation*}
V^{c_{i}}(x):=V^{S}\left(x, c_{i}\right) \tag{10}
\end{equation*}
$$

So we have the inequalities

$$
V^{c_{i}}(x) \geq V^{c_{i+1}}(x) \geq \ldots \geq V^{c_{n}}(x)=V^{\bar{c}}(x) \geq 0
$$

where $V^{\bar{c}}(x)=V^{\{\bar{c}\}}(x, \bar{c})$ as defined in (7).
The Hamilton-Jacobi-Bellman equation associated to 10 is given by

$$
\begin{equation*}
\max \left\{\mathcal{L}^{c_{i}}\left(V^{c_{i}}(x)\right), V^{c_{i+1}}(x)-V^{c_{i}}(x)\right\}=0 \text { for } x \geq 0 \text { and } i=1, \ldots, n-1 \tag{11}
\end{equation*}
$$

As in the continuous case we have that $V^{c_{i}}$ is the viscosity solution of the corresponding HJB equation. Let us introduce the definition of a viscosity solution in the one-dimensional case.

Definition 3.2 (a) A locally Lipschitz function $\bar{u}:[0, \infty) \rightarrow \mathbb{R}$ is a viscosity supersolution of (11) at $x \in(0, \infty)$ if any twice continuously differentiable function $\varphi:[0, \infty) \rightarrow \mathbb{R}$ with $\varphi(x)=\bar{u}(x)$ such that $\bar{u}-\varphi$ reaches the minimum at $x$ satisfies

$$
\max \left\{\mathcal{L}^{c_{i}}(\varphi)(x), V^{c_{i+1}}(x)-\varphi(x)\right\} \leq 0
$$

The function $\varphi$ is called a test function for supersolution at $x$.
(b) A function $\underline{u}:[0, \infty) \rightarrow \mathbb{R}$ is a viscosity subsolution of (11) at $x \in(0, \infty)$ if any twice continuously differentiable function $\psi:[0, \infty) \rightarrow \mathbb{R}$ with $\psi(x)=\underline{u}(x)$ such that $\underline{u}-\psi$ reaches the maximum at $x$ satisfies

$$
\max \left\{\mathcal{L}^{c_{i}}(\psi)(x), V^{c_{i+1}}(x)-\psi(x)\right\} \geq 0
$$

The function $\psi$ is called a test function for subsolution at $x$.
(c) A function $u:[0, \infty) \rightarrow \mathbb{R}$ which is both a supersolution and subsolution at $x \in[0, \infty)$ is called a viscosity solution of (11) at $x$.

The following characterization theorem is the analogue of Theorem 3.3 for finite sets; the proof is similar and simpler than the one in the continuous case.

Theorem 3.5 The optimal value function $V^{c_{i}}(x)$ for $1 \leq i<n$ is the unique viscosity solution of the associated HJB equation (11) with boundary condition $V^{c_{i}}(0)=0$ and $\lim _{x \rightarrow \infty} V^{c_{i}}(x)=$ $\bar{c} / q$.

We also have the alternative characterization theorem.
Theorem 3.6 The optimal value function $V^{c_{i}}(x)$ for $1 \leq i<n$ is the smallest viscosity supersolution of the the associated HJB equation (11) with boundary condition 0 at $x=0$ and limit greater than or equal to $\bar{c} / q$ as $x$ goes to infinity.

Remark 3.4 The function $V^{c_{n}}$ has the closed formula given by (7) for $c=c_{n}$. By the previous theorem, once $V^{c_{i+1}}$ is known, the optimal value function $V^{c_{i}}$ can be obtained recursively as the solution of the obstacle problem of finding the smallest viscosity supersolution of the equation $\mathcal{L}^{c_{i}}=0$ above the obstacle $V^{c_{i+1}}$.

## 4 Convergence of the optimal value functions from the discrete to the continuous case

In this section we prove that the optimal value functions of finite ratcheting strategies approximate the optimal value function of the continuous case as the mesh size of the finite sets goes to zero.

Consider for $n \geq 0$, a sequence of sets $\mathcal{S}^{n}$ (with $k_{n}$ elements) of the form

$$
\mathcal{S}^{n}=\left\{c_{1}^{n}=\underline{c}<c_{2}^{n}<\cdots<c_{k_{n}}^{n}=\bar{c}\right\} .
$$

satisfying $\mathcal{S}^{0}=\{\underline{c}, \bar{c}\}, \mathcal{S}^{n} \subset \mathcal{S}^{n+1}$ and mesh-size $\delta\left(\mathcal{S}^{n}\right):=\max _{k=2, k_{n}}\left(c_{k}^{n}-c_{k-1}^{n}\right) \searrow 0$ as $n$ goes to infinity.

Let us extend the definition of $V^{\mathcal{S}^{n}}$ to the function $V^{n}:[\underline{c}, \infty) \times[0, \bar{c}] \rightarrow \mathbb{R}$, as

$$
\begin{equation*}
V^{n}(x, c)=V^{\mathcal{S}^{n}}\left(x, \widetilde{c}^{n}\right) \tag{12}
\end{equation*}
$$

where

$$
\begin{equation*}
\widetilde{c}^{n}=\min \left\{c_{i}^{n} \in \mathcal{S}^{n}: c_{i}^{n} \geq c\right\} \tag{13}
\end{equation*}
$$

We will prove that $\lim _{n \rightarrow \infty} V^{n}(x, c)=V^{[c, c} \bar{c}(x, c)$ for any $(x, c) \in[0, \infty) \times[\underline{c}, \bar{c}]$ and we will study the uniform convergence of this limit.

Since $\left.V^{n} \leq V^{n+1} \leq V^{[c, c} \bar{c}\right]$, there exists the limit function

$$
\begin{equation*}
\bar{V}(x, c):=\lim _{n \rightarrow \infty} V^{n}(x, c) . \tag{14}
\end{equation*}
$$

Later on, we will show that $\left.\bar{V}=V^{[c, c} \bar{c}\right]$. Note that $\bar{V}(x, c)$ is non-increasing in $c$ with $\bar{V}(x, \bar{c})=$ $V(x, \bar{c})$, and non-decreasing in $x$ with $\lim _{x \rightarrow \infty} \bar{V}(x, c)=\bar{c} / q$. With the same proof the one for Proposition 6.1 of [2], we have the following proposition:

Proposition 4.1 The sequence $V^{n}$ converges uniformly to $\bar{V}$.
With this, we can obtain the main result of this section.
Theorem 4.2 The function $\bar{V}$ defined in (14) is the optimal value function $\left.V^{[c, ~}, \bar{c}\right]$.
Proof. Note that $\bar{V}(x, c)$ is a limit of value functions of admissible strategies, so in order to satisfy the assumptions of Theorem 3.4, it remains to see that $\bar{V}$ is a viscosity supersolution of (8) at any point $\left(x_{0}, c_{0}\right)$ with $x_{0}>0 . \partial_{c} \bar{V}\left(x_{0}, c_{0}\right) \leq 0$ in the viscosity sense because $\bar{V}$ is non-increasing in $c$; so it is sufficient to show that $\mathcal{L}^{c_{0}}(\bar{V})\left(x_{0}, c_{0}\right) \leq 0$ in the viscosity sense. Let $\varphi$ be a test function for viscosity supersolution of (8) at $\left(x_{0}, c_{0}\right)$, i.e. a (2,1)-differentiable function $\varphi$ with

$$
\begin{equation*}
\bar{V}(x, c) \geq \varphi(x, c) \text { and } \bar{V}\left(x_{0}, c_{0}\right)=\varphi\left(x_{0}, c_{0}\right) \tag{15}
\end{equation*}
$$

In order to prove that $\mathcal{L}^{c}(\varphi)\left(x_{0}, c_{0}\right) \leq 0$, consider now, for $\gamma>0$ small enough,

$$
\varphi_{\gamma}(x, c)=\varphi(x, c)-\gamma\left(x-x_{0}\right)^{4}
$$

Given $n \geq 1$, let us consider $\widetilde{c}_{0}^{n}$ as defined in 13),

$$
\begin{gathered}
a_{n}^{\gamma}:=\min \left\{V^{n}\left(x, \widetilde{c}_{0}^{n}\right)-\varphi_{\gamma}\left(x, \widetilde{c}_{0}^{n}\right): x \in\left[0, x_{0}+1\right]\right\}, \\
x_{n}^{\gamma}:=\arg \min \left\{V^{n}\left(x, \widetilde{c}_{0}^{n}\right)-\varphi_{\gamma}\left(x, \widetilde{c}_{0}^{n}\right): x \in\left[0, x_{0}+1\right]\right\},
\end{gathered}
$$

and

$$
b_{n}^{\gamma}:=\max \left\{\bar{V}\left(x, \widetilde{c}_{0}^{n}\right)-V^{n}\left(x, \widetilde{c}_{0}^{n}\right): x \in\left[0, x_{0}+1\right]\right\} .
$$

Since $\widetilde{c}_{0}^{n} \searrow c_{0}$ and, from Proposition 4.1, $\lim _{n \rightarrow \infty} a_{n}^{\gamma}=0$ and $\lim _{n \rightarrow \infty} b_{n}^{\gamma}=0$, we also have that $\lim _{n \rightarrow \infty} x_{n}^{\gamma}=x_{0}$ because

$$
\begin{aligned}
0 & =V^{n}\left(x_{n}^{\gamma}, \widetilde{c}_{0}^{n}\right)-\left(\varphi_{\gamma}\left(x_{n}^{\gamma}, \widetilde{c}_{0}^{n}\right)+a_{n}^{\gamma}\right) \\
& =\left(V^{n}\left(x_{n}^{\gamma}, \widetilde{c}_{0}^{n}\right)-\bar{V}\left(x_{n}^{\gamma}, \widetilde{c}_{0}^{n}\right)\right)+\left(\bar{V}\left(x_{n}^{\gamma}, \widetilde{c}_{0}^{n}\right)-\varphi_{\gamma}\left(x_{n}^{\gamma}, \widetilde{c}_{0}^{n}\right)\right)-a_{n}^{\gamma} \\
& \geq-b_{n}^{\gamma}+0-a_{n}^{\gamma}+\gamma\left(x_{n}^{\gamma}-x_{0}\right)^{4}
\end{aligned}
$$

and then

$$
\left(x_{n}^{\gamma}-x_{0}\right)^{4} \leq \frac{a_{n}^{\gamma}+b_{n}^{\gamma}}{\gamma} \rightarrow 0 \text { as } n \rightarrow \infty
$$

Note that $\bar{\varphi}^{n}(\cdot)=\varphi_{\gamma}\left(\cdot, \widetilde{c}_{0}^{n}\right)+a_{n}^{\gamma}$ is a test function for viscosity supersolution of $V^{n}\left(\cdot, \widetilde{c}_{0}^{n}\right)$ in Equation (11) at the point $x_{n}^{\gamma}$ because

$$
\varphi_{\gamma}\left(x_{n}^{\gamma}, \widetilde{c}_{0}^{n}\right)+a_{n}^{\gamma}=V^{n}\left(x_{n}^{\gamma}, \widetilde{c}_{0}^{n}\right) \text { and } \varphi_{\gamma}\left(x, \widetilde{c}_{0}^{n}\right)+a_{n}^{\gamma} \leq V^{n}\left(x, \widetilde{c}_{0}^{n}\right) \text { for } x \in\left[0, x_{0}+1\right] .
$$

And so

$$
\mathcal{L}^{\widetilde{c}_{0}^{n}}\left(\varphi_{\gamma}\right)\left(x_{n}^{\gamma}, \widetilde{c}_{0}^{n}\right)=\mathcal{L}^{\widetilde{c}_{0}^{n}}\left(\bar{\varphi}^{n}\right)\left(x_{n}^{\gamma}\right)+q a_{n}^{\gamma} \leq q a_{n}^{\gamma}
$$

Since $\left(x_{n}^{\gamma}, c_{n}\right) \rightarrow\left(x_{0}, c_{0}\right), \bar{\varphi}^{n}(\cdot)=\varphi_{\gamma}\left(\cdot, \widetilde{c}_{0}^{n}\right)+a_{n}^{\gamma} \rightarrow \varphi_{\gamma}\left(\cdot, c_{0}\right)$ as $n \rightarrow \infty$ and $\varphi_{\gamma}$ is (2,1)differentiable, one gets

$$
\mathcal{L}^{c_{0}}\left(\varphi_{\gamma}\right)\left(x_{0}, c_{0}\right)=\lim _{n \rightarrow \infty} \mathcal{L}^{\widetilde{c}_{0}^{n}}\left(\bar{\varphi}^{n}\right)\left(x_{n}^{\gamma}\right) \leq 0 .
$$

Finally, as

$$
\partial_{x} \varphi_{\gamma}\left(x_{0}, c_{0}\right)=\partial_{x} \varphi\left(x_{0}, c_{0}\right) \text { and } \partial_{x x} \varphi_{\gamma}\left(x_{0}, c_{0}\right)=\partial_{x x} \varphi\left(x_{0}, c_{0}\right)
$$

and $\varphi_{\gamma} \nearrow \varphi$ as $\gamma \searrow 0$, we obtain that $\mathcal{L}^{c_{0}}(\varphi)\left(x_{0}, c_{0}\right) \leq 0$ and the result follows.

## 5 The optimal strategies

We show first that, regardless whether $S$ is finite or an interval with max $S=\bar{c}$, the optimal strategy for sufficiently small $\bar{c}$ is to immediately start paying dividends at the maximum rate $\bar{c}$.

Proposition 5.1 If $\bar{c} \leq q \sigma^{2} /(2 \mu)$, then $V(x, c)=V^{\{\bar{c}\}}(x, \bar{c})$ for any $(x, c) \in[0, \infty) \times S$.
Proof. If we call $W(x, c):=V^{\{\bar{c}\}}(x, \bar{c})$, then we know that $\mathcal{L}^{\bar{c}}(W)(x, c)=0$. Since $W(0, \bar{c})=$ $0, \lim _{x \rightarrow \infty} W(x, c)=\bar{c} / q$ and $\partial_{c} W(x, c)=0$, then by Theorem 3.3 and Theorem3.5 it is enough to prove that $\mathcal{L}^{c}(W)(x, c) \leq 0$ for $c \in S$. But, by (7) and (5)

$$
\begin{aligned}
\mathcal{L}^{c}(W)(x, c) & =\mathcal{L}^{\bar{c}}(W)(x, c)+(\bar{c}-c)\left(\partial_{x} W(x, c)-1\right) \\
& =(\bar{c}-c)\left(-\frac{\bar{c}}{q} \theta_{2}(\bar{c}) e^{\theta_{2}(\bar{c}) x}-1\right) \\
& \leq(\bar{c}-c)\left(-\frac{c}{q} \theta_{2}(\bar{c})-1\right) \\
& \leq 0
\end{aligned}
$$

for $c \leq \bar{c} \leq \frac{q \sigma^{2}}{2 \mu}$.
Remark 5.1 The proof of the previous proposition also shows that if there exists a $d \in S \backslash\{\bar{c}\}$ and $\varepsilon>0$ such that $V(x, d)=V^{\{\bar{c}\}}(x, \bar{c})$ for $x \in[0, \varepsilon]$, then $\bar{c} \leq \frac{q \sigma^{2}}{2 \mu}$ and so $V(x, c)=V^{\{\bar{c}\}}(x, \bar{c})$ for any $(x, c) \in[0, \infty) \times S$.

Let us first address the case of $S=\left\{c_{1}, c_{2}, \ldots ., c_{n}\right\}$ with $0 \leq c_{1}<c_{2}<\ldots .<c_{n}=\bar{c}$. We introduce the concept of strategies with a threshold structure for each level $c_{i} \in S$ and prove that there exists an optimal dividend payment strategy and has this form. Later we extend the concept of strategies with this type of structure to the case $S=[\underline{c}, \bar{c}]$ by means of a curve in the state space $[0, \infty) \times[\underline{c}, \bar{c}]$ and look for the curve which maximizes the expected discounted cumulative dividends.

### 5.1 Optimal strategies for finite sets

Take $S=\left\{c_{1}, c_{2}, \ldots, c_{n}\right\}$ with $0 \leq c_{1}<c_{2}<\ldots .<c_{n}=\bar{c}$. Since for $i \leq n-1$, the optimal value function $V^{c_{i}}$ is a viscosity solution of (11), there are values of $x$ where $V^{c_{i}}(x)=V^{c_{i+1}}(x)$ and values of $x$ where $\mathcal{L}^{c_{i}}\left(V^{c_{i}}\right)(x)=0$. We look for the simplest dividend payment strategies, those whose value functions are solutions of $\mathcal{L}^{c_{i}}=0$ for $x \in\left[0, z\left(c_{i}\right)\right)$ and $V^{c_{i}}=V^{c_{i+1}}$ for $x \in\left[z\left(c_{i}\right), \infty\right)$ with some $z\left(c_{i}\right) \geq 0$. We will show in this subsection that the optimal value function comes from such types of strategies. More precisely, take $\widetilde{S}=S \backslash\left\{c_{n}\right\}$ and a function $z: \widetilde{S} \rightarrow[0, \infty)$; we define a threshold strategy by backward recursion, it is a stationary strategy (which depends on both the current surplus $x$ and the implemented dividend rate $c_{i} \in S$ )

$$
\begin{equation*}
\pi^{z}=\left(C_{x, c_{i}}\right)_{\left(x, c_{i}\right) \in[0, \infty) \times S} \text { where } C_{x, c_{i}} \in \Pi_{x, c_{i}}^{S} \tag{16}
\end{equation*}
$$

as follows:

- If $i=n$, pay dividends with rate $c_{n}=\bar{c}$ up to the time of ruin, that is $\left(C_{x, c_{n}}\right)_{t}=\bar{c}$.
- If $1 \leq i<n$ and $x \geq z\left(c_{i}\right)$ follow $C_{x, c_{i+1}} \in \Pi_{x, c_{i+1}}^{S}$.
- If $1 \leq i<n$ and $x<z\left(c_{i}\right)$ pay dividends with rate $c_{i}$ as long as the surplus is less than $z\left(c_{i}\right)$ up to the ruin time; if the current surplus reaches $z\left(c_{i}\right)$ before the time of ruin, follow $C_{x, c_{i+1}} \in \Pi_{x, c_{i+1}}^{S}$. More precisely

$$
\left(C_{x, c_{i}}\right)_{t}=c_{i} I_{t \leq \tau \wedge \widehat{\tau}}+\left(C_{X_{\widehat{\tau}}, c_{i+1}}\right)_{t} I_{\widehat{\tau} \leq t<\tau},
$$

where $\widehat{\tau}$ is the first time at which the surplus reaches $z\left(c_{i}\right)$.

Let us call the value $z\left(c_{i}\right)$ the threshold at dividend rate level $c_{i}$ and $z: \widetilde{S} \rightarrow[0, \infty)$ the threshold function. The value function of the stationary strategy $\pi^{z}$ is defined as

$$
\begin{equation*}
W^{z}\left(x, c_{i}\right):=J\left(x ; C_{x, c_{i}}\right) \tag{17}
\end{equation*}
$$

Note that $W^{z}\left(x, c_{i}\right)$ only depends on $z\left(c_{k}\right)$ for $i \leq k<n, W^{z}\left(0, c_{i}\right)=0$ and $W^{z}\left(x, c_{i}\right)=V^{c_{n}}(x)$ for $x \geq \max \left\{z\left(c_{k}\right): i<k<n\right\}$.

Proposition 5.2 We have the following recursive formula for $W^{z}$ :

$$
\begin{aligned}
W^{z}\left(x, c_{n}\right) & =\frac{c_{n}}{q}\left(1-e^{\theta_{2}\left(c_{n}\right) x}\right), \\
W^{z}\left(x, c_{i}\right) & = \begin{cases}W^{z}\left(x, c_{i+1}\right) & \text { if } x \geq z\left(c_{i}\right) \\
\frac{c_{i}}{q}\left(1-e^{\theta_{2}\left(c_{i}\right) x}\right)+a^{z}\left(c_{i}\right)\left(e^{\theta_{1}\left(c_{i}\right) x}-e^{\theta_{2}\left(c_{i}\right) x}\right) & \text { if } x<z\left(c_{i}\right)\end{cases}
\end{aligned}
$$

for $i<n$, where

$$
a^{z}\left(c_{i}\right):=\frac{W^{z}\left(z\left(c_{i}\right), c_{i+1}\right)-\frac{c_{i}}{q}\left(1-e^{\theta_{2}\left(c_{i}\right) z\left(c_{i}\right)}\right)}{e^{\theta_{1}\left(c_{i}\right) z\left(c_{i}\right)}-e^{\theta_{2}\left(c_{i}\right) z\left(c_{i}\right)}} .
$$

Proof. We have that $\mathcal{L}^{c_{i}}\left(W^{z}\right)\left(x, c_{i}\right)=0$ for $x \in\left(0, z\left(c_{i}\right)\right)$ because the stationary strategy $\pi^{z}$ pays $c_{i}$ when the current surplus is in $\left(0, z\left(c_{i}\right)\right)$. Also $W^{z}\left(0, c_{i}\right)=0$ because ruin is immediate at $x=0$, and by definition $W^{z}\left(z\left(c_{i}\right), c_{i}\right)=W^{z}\left(z\left(c_{i}\right), c_{i+1}\right)$. From (6), we get the result.

Let us now look for the maximum of the value functions $W^{z}\left(x, c_{i}\right)$ among all the possible threshold functions $z: \widetilde{S} \rightarrow[0, \infty)$, and denote by $z^{*}$ the optimal threshold function. From Proposition 5.1. $z^{*}=0$ for $\bar{c}=c_{n} \leq q \sigma^{2} /(2 \mu)$, so that from now on we only consider the case $c_{n}>q \sigma^{2} /(2 \mu)$.

Since the function $W^{z}\left(x, c_{n}\right)$ is known, there are two ways to solve this optimization problem (using a backward recursion). We will study the problem using both of them.

1. The first approach consists of seeing the optimization problem as a sequence of $n-1$ one-dimensional optimization problems, that is obtaining the maximum $a^{z}\left(c_{i}\right)$ for $i=$ $n-1, \ldots, 1$. If $W^{z^{*}}\left(x, c_{k}\right)$ and $z^{*}\left(c_{k}\right)$ are known for $k=i+1, \ldots, n$, then from Proposition 5.2 we can obtain

$$
\begin{equation*}
z^{*}\left(c_{i}\right)=\min \left(\arg \max _{y \in[0, \infty)} \frac{W^{z^{*}}\left(y, c_{i+1}\right)-\frac{c_{i}}{q}\left(1-e^{\theta_{2}\left(c_{i}\right) y}\right)}{e^{\theta_{1}\left(c_{i}\right) y}-e^{\theta_{2}\left(c_{i}\right) y}}\right) . \tag{18}
\end{equation*}
$$

Note that

$$
\lim _{y \rightarrow \infty} \frac{W^{z^{*}}\left(y, c_{i+1}\right)-\frac{c_{i}}{q}\left(1-e^{\theta_{2}\left(c_{i}\right) y}\right)}{e^{\theta_{1}\left(c_{i}\right) y}-e^{\theta_{2}\left(c_{i}\right) y}}=0
$$

because $\lim _{y \rightarrow \infty} W^{z^{*}}\left(y, c_{i+1}\right)=\frac{c_{n}}{q}$, so $z^{*}\left(c_{i}\right)$ exists.
2. As a second approach, one can view the optimization problem as a backward recursion of obstacle problems (see Remark 3.4 ). If $W^{z^{*}}\left(x, c_{k}\right)$ and $z^{*}\left(c_{k}\right)$ are known for $k=$ $i+1, \ldots, n$, we look for the smallest solution $U^{*}$ of the equation $\mathcal{L}^{c_{i}}(U)=0$ in $[0, \infty)$ with boundary condition $U(0)=0$ above $W^{z^{*}}\left(\cdot, c_{i+1}\right)$. Then

$$
\begin{equation*}
z^{*}\left(c_{i}\right)=\min \left\{y>0: U^{*}(y)=W^{z^{*}}\left(y, c_{i+1}\right)\right\} . \tag{19}
\end{equation*}
$$

By (6), the solutions $U$ of the equation $\mathcal{L}^{c_{i}}(U)=0$ in $[0, \infty)$ with boundary condition $U(0)=0$ are of the form

$$
U_{a}(x)=\frac{c_{i}}{q}\left(1-e^{\theta_{2}\left(c_{i}\right) x}\right)+a\left(e^{\theta_{1}\left(c_{i}\right) x}-e^{\theta_{2}\left(c_{i}\right) x}\right)
$$

Hence $U_{a}(x)$ is increasing in $a$ and $\lim _{a \rightarrow \infty} U_{a}(x)=\infty$ for $x>0$, and so there exists an $a_{i}^{*}>0$ such that

$$
U^{*}=U_{a_{i}^{*}}=\min \left\{U_{a}: U_{a}(x) \geq W^{z^{*}}\left(x, c_{i+1}\right) \text { for all } x \geq 0\right\}
$$

because $\lim _{x \rightarrow \infty} U_{0}(x)=\frac{c_{i}}{q}<\frac{c_{n}}{q}=\lim _{x \rightarrow \infty} W^{z^{*}}\left(x, c_{i+1}\right)$.
Remark 5.2 In the second approach, we can see $z^{*}\left(c_{i}\right)$ as the smallest $x>0$ such that $U_{a^{*}}$ and $W^{z^{*}}\left(\cdot, c_{i+1}\right)$ coincide; more precisely $U_{a^{*}}\left(z^{*}\left(c_{i}\right)\right)=W^{z^{*}}\left(z^{*}\left(c_{i}\right), c_{i+1}\right), U_{a^{*}}(x) \geq W^{z^{*}}\left(x, c_{i+1}\right)$ for $x>0$ and $U_{a^{*}}(x)>W^{z^{*}}\left(x, c_{i+1}\right)$ for $x \in\left(0, z^{*}\left(c_{i}\right)\right)$. Note that $U_{a^{*}}^{\prime}\left(z^{*}\left(c_{i}\right)\right)=\partial_{x} W^{z^{*}}\left(z^{*}\left(c_{i}\right), c_{i+1}\right)$ and $U_{a^{*}}(\cdot)-W^{z^{*}}\left(\cdot, c_{i+1}\right)$ is locally convex at $z^{*}\left(c_{i}\right)$. By the recursive construction, this implies that $W^{z^{*}}\left(x, c_{i}\right)$ is infinitely continuously differentiable at all $x \in[0, \infty) \backslash\left\{z^{*}\left(c_{k}\right): k=i, \ldots, n-1\right\}$ and continuously differentiable at the points $z^{*}\left(c_{k}\right)$ for $k=i, \ldots, n-1$.

Lemma $5.3 U_{0}(x)$ is an increasing concave function. If $a>0, U_{a}(x)$ is increasing, and is concave in $\left(-\infty, y_{0}\right)$ and convex in $\left(y_{0}, \infty\right)$ with

$$
y_{0}:=\frac{\log \left(\left(\frac{c}{q}+a\right) \theta_{2}(c)^{2}\right)-\log \left(a \theta_{1}(c)^{2}\right)}{\theta_{1}(c)-\theta_{2}(c)}
$$

In the case $c \leq \mu$, we have that $y_{0}>0$; in the case $c>\mu$, we have that $y_{0} \leq 0$ if and only if

$$
0<a<\frac{c}{q} \frac{\theta_{2}(c)^{2}}{\left(\theta_{1}(c)^{2}-\theta_{2}(c)^{2}\right)} .
$$

Proof. We have that

$$
\partial_{x} U_{a}(x)=-\left(\frac{c}{q}+a\right) \theta_{2}(c) e^{\theta_{2}(c) x}+a \theta_{1}(c) e^{\theta_{1}(c) x}>0
$$

and

$$
\partial_{x x} U_{a}(x)=-\left(\frac{c}{q}+a\right) \theta_{2}(c)^{2} e^{\theta_{2}(c) x}+a \theta_{1}(c)^{2} e^{\theta_{1}(c) x} \geq 0
$$

if and only if $x \geq y_{0}$. The result follows from Definition 5
In the next theorem, we show that there exists an optimal strategy and it is of threshold type.
Theorem 5.4 If $z^{*}$ is the optimal threshold function, then $W^{z^{*}}\left(x, c_{i}\right)$ is the optimal function $V^{c_{i}}(x)$ defined in (2) for $i=1, \ldots, n$.

Proof. By definition $W^{z^{*}}\left(\cdot, c_{n}\right)=V^{c_{n}}$. Assuming that $W^{z^{*}}\left(\cdot, c_{i+1}\right)=V^{c_{i+1}}$ for $i=n-$ $1, \ldots, 1$, by Theorem 3.5, it is enough to prove that $W^{z^{*}}\left(\cdot, c_{i}\right)$ is a viscosity solution of 11.). Since by construction $V^{c_{i+1}}-W^{z^{*}}\left(\cdot, c_{i}\right) \leq 0$, it remains to be seen that $\mathcal{L}^{c_{i}}\left(W^{z^{*}}\right)\left(x, c_{i}\right) \leq 0$ for $x \geq z^{*}\left(c_{i}\right)$. By Remark 5.2, $W^{z^{*}}\left(\cdot, c_{i}\right)$ is continuously differentiable and it is piecewise infinitely differentiable in open intervals in which it solves $\mathcal{L}^{c_{j}}\left(W^{z^{*}}\right)\left(x, c_{i}\right)=0$ for some $j \geq i$. By the definition of a viscosity solution, it is enough to prove the result in these open intervals. For $x$ in these open intervals,

$$
\mathcal{L}^{c_{i}}\left(W^{z^{*}}\right)\left(x, c_{i}\right)=\mathcal{L}^{c_{j}}\left(W^{z^{*}}\right)\left(x, c_{i}\right)+\left(c_{i}-c_{j}\right)\left(1-\partial_{x} W^{z^{*}}\left(x, c_{i}\right)\right) \leq 0
$$

if and only if $\partial_{x} W^{z^{*}}\left(x, c_{i}\right) \leq 1$. There exists $\delta>0$ and some $j>i$ such that $\mathcal{L}^{c_{j}}\left(W^{z^{*}}\right)\left(x, c_{i}\right)=0$ in $\left(z^{*}\left(c_{i}\right), z^{*}\left(c_{i}\right)+\delta\right)$ and then

$$
\mathcal{L}^{c_{j}}\left(W^{z^{*}}\right)\left(z^{*}\left(c_{i}\right)^{+}, c_{i}\right)=0, \mathcal{L}^{c_{i}}\left(W^{z^{*}}\right)\left(z^{*}\left(c_{i}\right)^{-}, c_{i}\right)=0
$$

$$
\begin{aligned}
0= & \mathcal{L}^{c_{i}}\left(W^{z^{*}}\right)\left(z^{*}\left(c_{i}\right)^{-}, c_{i}\right)-\mathcal{L}^{c_{j}}\left(W^{z^{*}}\right)\left(z^{*}\left(c_{i}\right)^{+}, c_{i}\right) \\
= & \frac{\sigma^{2}}{2}\left(\partial_{x x} W^{z^{*}}\left(z^{*}\left(c_{i}\right)^{-}, c_{i}\right)-\partial_{x x} W^{z^{*}}\left(z^{*}\left(c_{i}\right)^{+-}, c_{i}\right)\right) \\
& +\left(c_{i}-c_{j}\right)\left(1-\partial_{x} W^{z^{*}}\left(z^{*}\left(c_{i}\right), c_{i}\right)\right) .
\end{aligned}
$$

By Remark 5.2, $\partial_{x x} W^{z^{*}}\left(z^{*}\left(c_{i}\right)^{-}, c_{i}\right)-\partial_{x x} W^{z^{*}}\left(z^{*}\left(c_{i}\right)^{+-}, c_{i}\right) \geq 0$ and $c_{i}-c_{j}<0$, so we conclude that $\left.\partial_{x} W^{z^{*}}\left(z^{*}\left(c_{i}\right), c_{i}\right)\right) \leq 1$.

If $i=n-1, W^{z^{*}}\left(\cdot, c_{n-1}\right)=W^{z^{*}}\left(\cdot, c_{n}\right)$ for $x \geq z^{*}\left(c_{n-1}\right)$, by Remark 5.3, $W^{z^{*}}\left(\cdot, c_{n}\right)$ is concave and so $W^{z^{*}}\left(x, c_{n}\right) \leq W^{z^{*}}\left(z^{*}\left(c_{n-1}\right), c_{n}\right) \leq 1$ and we have the result.

We need to prove that $\partial_{x} W^{z^{*}}\left(x, c_{i}\right)=\partial_{x} W^{z^{*}}\left(x, c_{i+1}\right) \leq 1$ for $x \geq z^{*}\left(c_{i}\right)$. By induction hypothesis, we know that $\partial_{x} W^{z^{*}}\left(x, c_{i+1}\right)=\partial_{x} V^{c_{i+1}} \leq 1$ for $x \geq z^{*}\left(c_{i+1}\right)$. In the case that $z^{*}\left(c_{i}\right) \geq z^{*}\left(c_{i+1}\right)$, it is straightforward; in the case that $z^{*}\left(c_{i}\right)<z^{*}\left(c_{i+1}\right)$, it is enough to prove it in the interval $\left(z^{*}\left(c_{i}\right), z^{*}\left(c_{i+1}\right)\right)$. But $\partial_{x} W^{z^{*}}\left(z^{*}\left(c_{i}\right), c_{i}\right) \leq 1, \partial_{x} W^{z^{*}}\left(z^{*}\left(c_{i+1}\right), c_{i}\right)=$ $\partial_{x} W^{z^{*}}\left(z^{*}\left(c_{i+1}\right), c_{i+1}\right) \leq 1$, and by Lemma $5.3 \partial_{x} W^{z^{*}}\left(x, c_{i}\right)$ is either increasing, or decreasing, or decreasing and then increasing in the interval $\left(z^{*}\left(c_{i}\right), z^{*}\left(c_{i+1}\right)\right)$, so that we have the result.

Taking the derivative in with respect to $y$, we get implicit equations for the optimal threshold strategy.

Proposition $5.5 z^{*}\left(c_{i}\right)$ satisfies the implicit equation

$$
\begin{aligned}
0 & =\frac{c_{i}}{q} \theta_{2}\left(c_{i}\right) e^{\theta_{2}\left(c_{i}\right) y}\left(e^{\theta_{1}\left(c_{i}\right) y}-e^{\theta_{2}\left(c_{i}\right) y}\right)-\frac{c_{i}}{q}\left(1-e^{\theta_{2}\left(c_{i}\right) y}\right)\left(\theta_{1}\left(c_{i}\right) e^{\theta_{1}\left(c_{i}\right) y}-\theta_{2}\left(c_{i}\right) e^{\theta_{2}\left(c_{i}\right) y}\right) \\
& +\partial_{x} W^{z^{*}}\left(y, c_{i+1}\right)\left(e^{\theta_{1}\left(c_{i}\right) y}-e^{\theta_{2}\left(c_{i}\right) y}\right)-W^{z^{*}}\left(y, c_{i+1}\right)\left(\theta_{1}\left(c_{i}\right) e^{\theta_{1}\left(c_{i}\right) y}-\theta_{2}\left(c_{i}\right) e^{\theta_{2}\left(c_{i}\right) y}\right)
\end{aligned}
$$

for $i=n-1, \ldots, 1$.
Remark 5.3 Given $z: \widetilde{S} \rightarrow[0, \infty)$, we have defined in 16 a threshold strategy $\pi^{z}=$ $\left(C_{x, c_{i}}\right)_{\left(x, c_{i}\right) \in[0, \infty) \times S}$, where $C_{x, c_{i}} \in \Pi_{x, c_{i}}^{S}$ for $i=1, \ldots, n$. We can extend this threshold strategy to

$$
\begin{equation*}
\tilde{\pi}^{z}=\left(C_{x, c}\right)_{(x, c) \in[0, \infty) \times\left[c_{1}, c_{n}\right]} \text { where } C_{x, c} \in \Pi_{x, c}^{S} \tag{20}
\end{equation*}
$$

as follows:

- If $c \in\left(c_{i}, c_{i+1}\right)$ and $x<z\left(c_{i}\right)$, pay dividends with rate $c$ while the current surplus is less than $z\left(c_{i}\right)$ up to the time of ruin. If the current surplus reaches $z\left(c_{i}\right)$ before the time of ruin, follow $C_{z\left(c_{i}\right), c_{i+1}} \in \Pi_{x, c_{i+1}}^{S}$.
- If $c \in\left(c_{i}, c_{i+1}\right)$ and $x \geq z\left(c_{i}\right)$ for $1 \leq i<n$, follow $C_{x, c_{i+1}} \in \Pi_{x, c_{i+1}}^{S}$.

The value function of the stationary strategy $\widetilde{\pi}^{z}$ is defined as

$$
\begin{equation*}
J^{\tilde{\pi}^{z}}(x, c):=J\left(x ; C_{x, c}\right):[0, \infty) \times\left[c_{1}, c_{n}\right] \rightarrow \mathbb{R} \tag{21}
\end{equation*}
$$

### 5.2 Curve strategies and the optimal curve strategy

As it is typical for these type of problems, the way in which the optimal value function $V(x, c)$ solves the HJB equation (8) suggests that the state space $[0, \infty) \times[\underline{c}, \bar{c}]$ is partitioned into two regions: a non-change dividend region $\mathcal{N C}^{*}$ in which the dividends are paid at constant rate and a change dividend region $\mathcal{C H}{ }^{*}$ in which the rate of dividends increases. Roughly speaking, the region $\mathcal{N C}^{*}$ consists of the points in the state space where $\mathcal{L}^{c}(V)=0$ and $\partial_{c} V<0$ and $\mathcal{C H}{ }^{*}$ consists of the points where $\partial_{c} V=0$. We introduce a family of stationary strategies (or limit of stationary strategies) where the change and non-change dividend payment regions are connected and split by a free boundary curve. This family of strategies is the analogue to the threshold strategies for finite $S$ introduced in Section 5.1 .

Let us consider the set
$\mathcal{B}=\left\{\zeta\right.$ s.t. $\zeta:[\underline{c}, \bar{c}) \rightarrow[0, \infty)$ is Riemann integrable and càdlàg with $\left.\lim _{c \rightarrow \bar{c}^{-}} \zeta(c)<\infty\right\}$.

In the first part of this subsection, we define the $\zeta$-value function $W^{\zeta}$ associated to a curve

$$
\mathcal{R}(\zeta)=\{(\zeta(c), c): c \in[\underline{c}, \bar{c})\} \subset[0, \infty) \times[\underline{c}, \bar{c})
$$

for $\zeta \in \mathcal{B}$, and we will see that, in some sense, $W^{\zeta}(x, c)$ is a (limit) value function of the strategy which pays dividends at constant rate in the case that $x<\zeta(c)$ and otherwise increases the rate of dividends. So the curve $\mathcal{R}(\zeta)$ splits the state space $[0, \infty) \times[\underline{c}, \bar{c}]$ into two connected regions: $\mathcal{N C}(\zeta)=\{(x, c) \in[0, \infty) \times[\underline{c}, \bar{c}]: x<\zeta(c)\}$ where dividends are paid with constant rate, and $\mathcal{C H}(\zeta)=\{(x, c) \in[0, \infty) \times[\underline{c}, \bar{c}]: x \geq \zeta(c)\}$ where the dividend rate increases. In the second part of the subsection we then will look for the $\zeta_{0} \in \mathcal{B}$ that maximizes the $\zeta$-value function $W^{\zeta}$, using calculus of variations.

Let us consider the following auxiliary functions $b_{0}, b_{1}:(0, \infty) \times[\underline{c}, \bar{c}] \rightarrow \mathbb{R}$

$$
\begin{align*}
b_{0}(x, c) & :=\frac{-\frac{1}{q}\left(1-e^{\theta_{2}(c) x}\right)+\frac{c}{q} \theta_{2}^{\prime}(c) e^{\theta_{2}(c) x} x}{e^{\theta_{1}(c) x}-e^{\theta_{2}(c) x}},  \tag{23}\\
b_{1}(x, c) & :=\frac{\left(-\theta_{1}^{\prime}(c) e^{\theta_{1}(c) x}+\theta_{2}^{\prime}(c) e^{\theta_{2}(c) x}\right) x}{e^{\theta_{1}(c) x}-e^{\theta_{2}(c) x}} .
\end{align*}
$$

Both $b_{0}(x, c)$ and $b_{1}(x, c)$ are not defined in $x=0$, so we extend the definition as

$$
b_{0}(0, c)=\lim _{x \rightarrow 0^{+}} b_{0}(x, c):=\frac{c \theta_{2}^{\prime}(c)-\theta_{2}(c)}{q\left(\theta_{1}(c)-\theta_{2}(c)\right)}
$$

and

$$
b_{1}(0, c)=\lim _{x \rightarrow 0^{+}} b_{1}(x, c):=\frac{\theta_{2}^{\prime}(c)-\theta_{1}^{\prime}(c)}{\theta_{1}(c)-\theta_{2}(c)} .
$$

In order to define the $\zeta$-value function in the non-change region $\mathcal{N C}(\zeta)$, we will define and study in the next technical lemma the functions $H^{\zeta}$ and $A^{\zeta}$ for any $\zeta \in \mathcal{B}$.

Lemma 5.6 Given $\zeta \in \mathcal{B}$, the unique continuous function $H^{\zeta}:\{(x, c) \in[0, \infty) \times[\underline{c}, \bar{c}]: x \leq$ $\zeta(c)\} \rightarrow[0, \infty)$ which satisfies for any $c \in[\underline{c}, \bar{c})$ that

$$
\mathcal{L}^{c}\left(H^{\zeta}\right)(x, c)=0 \text { for } 0 \leq x \leq \zeta(c)
$$

with boundary conditions $H^{\zeta}(0, c)=0, H^{\zeta}(x, \bar{c})=V^{\{\bar{c}\}}(x, \bar{c})$ and $\partial_{c} H^{\zeta}(\zeta(c), c)=0$ at the points of continuity of $\zeta$ is given by

$$
\begin{equation*}
H^{\zeta}(x, c)=\frac{c}{q}\left(1-e^{\theta_{2}(c) x}\right)+A^{\zeta}(c)\left(e^{\theta_{1}(c) x}-e^{\theta_{2}(c) x}\right) \tag{24}
\end{equation*}
$$

where

$$
\begin{equation*}
A^{\zeta}(c)=-\int_{c}^{\bar{c}} e^{-\int_{c}^{t} b_{1}(\zeta(s), s) d s} b_{0}(\zeta(t), t) d t \tag{25}
\end{equation*}
$$

Moreover, $A^{\zeta}$ satisfies $A^{\zeta}(\bar{c})=0$, is differentiable and satisfies

$$
\begin{equation*}
\left(A^{\zeta}\right)^{\prime}(c)=b_{0}(\zeta(c), c)+b_{1}(\zeta(c), c) A^{\zeta}(c) \tag{26}
\end{equation*}
$$

at the points where $\zeta$ is continuous.
Proof. Since $\mathcal{L}^{c}\left(H^{\zeta}(x, c)\right)=0$ and $H^{\zeta}(0, c)=0$, we can write by (6)

$$
H^{\zeta}(x, c)=\frac{c}{q}\left(1-e^{\theta_{2}(c) x}\right)+A^{\zeta}(c)\left(e^{\theta_{1}(c) x}-e^{\theta_{2}(c) x}\right)
$$

where $A^{\zeta}(c)$ should be defined in such a way that $A^{\zeta}(\bar{c})=0$ (because $H^{\zeta}(x, \bar{c})=V^{\{\bar{c}\}}(x, \bar{c})$ ) and

$$
\begin{aligned}
0=\partial_{c} H^{\zeta}(\zeta(c), c)= & \frac{1}{q}\left(1-e^{\theta_{2}(c) \zeta(c)}\right)-\frac{c}{q} \theta_{2}^{\prime}(c) e^{\theta_{2}(c) \zeta(c)} \zeta(c)+A^{\zeta}(c)^{\prime}\left(e^{\theta_{1}(c) \zeta(c)}-e^{\theta_{2}(c) \zeta(c)}\right) \\
& +A^{\zeta}(c)\left(\theta_{1}^{\prime}(c) e^{\theta_{1}(c) \zeta(c)}-\theta_{2}^{\prime}(c) e^{\theta_{2}(c) \zeta(c)}\right) \zeta(c)
\end{aligned}
$$

at the points of continuity of $\zeta$. Hence,

$$
\begin{aligned}
\left(A^{\zeta}\right)^{\prime}(c) & =\frac{-\frac{1}{q}\left(1-e^{\theta_{2}(c) x}\right)+\frac{c}{q} \theta_{2}^{\prime}(c) e^{\theta_{2}(c) x} x}{e^{\theta_{1}(c) x}-e^{\theta_{2}(c) x}}+A^{\zeta}(c) \frac{\left(-\theta_{1}^{\prime}(c) e^{\theta_{1}(c) x}+\theta_{2}^{\prime}(c) e^{\theta_{2}(c) x}\right) x}{e^{\theta_{1}(c) x}-e^{\theta_{2}(c) x}}(*) \\
& =b_{0}(\zeta(c), c)+b_{1}(\zeta(c), c) A^{\zeta}(c)
\end{aligned}
$$

Solving this ODE with boundary condition $A^{\zeta}(\bar{c})=0$, we get the result.
Given $\zeta \in \mathcal{B}$, we define the $\zeta$-value function

$$
W^{\zeta}(x, c):= \begin{cases}H^{\zeta}(x, c) & \text { if } \quad(x, c) \in \mathcal{N C}(\zeta),  \tag{27}\\ H^{\zeta}(x, C(x, c)) & \text { if } \quad(x, c) \in \mathcal{C H}(\zeta),\end{cases}
$$

where $H^{\zeta}$ is defined in Lemma 5.6 and

$$
\begin{equation*}
C(x, c):=\max \{h \in[c, \bar{c}]: \zeta(d) \leq x \text { for } d \in[c, h)\} \tag{28}
\end{equation*}
$$

in the case that $x \geq \zeta(c)$ and $c \in[\underline{c}, \bar{c})$.
In the next propositions we will show that the $\zeta$-value function $W^{\zeta}$ is the value function of an extended threshold strategy in the case that $\zeta$ is a step function, and the limit of value functions of extended threshold strategies in the case that $\zeta \in \mathcal{B}$.

Proposition 5.7 Given $z: \widetilde{S} \rightarrow[0, \infty)$ and the corresponding extended threshold strategy $\widetilde{\pi}^{z}$ defined in Remark 5.3, let us consider the associated step function $\zeta \in \mathcal{B}$ defined as

$$
\zeta(c):=\sum_{i=1}^{n-1} z\left(c_{i}\right) I_{\left[c_{i}, c_{i+1}\right)}
$$

Then the stationary value function of the extended threshold strategy $\widetilde{\pi}^{z}$ is given by

$$
J_{\tilde{\pi}^{z}}(x, c)=W^{\zeta}(x, c) .
$$

Proof. The stationary value function is continuous and satisfies $\mathcal{L}^{c}\left(W^{\zeta}\right)(x, c)=0$ for $0 \leq x \leq \zeta(c), W^{\zeta}(0, c)=0, W^{\zeta}(x, \bar{c})=V^{\{\bar{c}\}}(x, \bar{c})$ and $\partial_{c} W^{\zeta}(\zeta(c), c)=0$ for $c \notin \widetilde{S}$. Also the right-hand derivatives $\partial_{c} W^{\zeta}\left(\zeta\left(c_{i}\right), c_{i}^{+}\right)=0$ for $i=1, \ldots, n-1$. So, by Lemma 5.6 , we obtain that $W^{\zeta}(x, c)=H^{\zeta}(x, c)$ if $x<\zeta(c)$. If $x \geq \zeta(c)$, the result follows from the definition of $\widetilde{\pi}^{z}$.

In the previous proposition we showed that in the case where $\zeta$ is the associated step function of $z$, the stationary strategy $\widetilde{\pi}^{z}$ consists of increasing immediately the divided rate from $c$ to $C(x, c)$ for $(x, c) \in \mathcal{C H}(\zeta)$, paying dividends at rate $c$ until either reaching the curve $\mathcal{R}(\zeta)$ or ruin (whatever comes first) for $(x, c) \in \mathcal{C}(\zeta)$, and paying dividends at rate $\bar{c}$ until the time of ruin for $c=\bar{c}$.

In the next proposition we show that for any $\zeta \in \mathcal{B}$, the $\zeta$-value function $W^{\zeta}$ is the limit of value functions of extended threshold strategies.

Proposition 5.8 Given $\zeta \in \mathcal{B}$, there exists a sequence of right-continuous step functions $\zeta_{n}$ : $[\underline{c}, \bar{c}) \rightarrow[0, \infty)$ such that $W^{\zeta_{n}}(x, c)$ converges uniformly to $W^{\zeta}(x, c)$.

Proof. Since $\zeta$ is a Riemann integrable càdlàg function, we can approximate it uniformly by right-continuous step functions. Namely, take a sequence of finite sets $\mathcal{S}^{k}=\left\{c_{1}^{k}, c_{2}^{k}, \cdots, c_{n_{k}}^{k}\right\}$ with $\underline{c}=c_{1}^{k}<c_{2}^{k}<\cdots<c_{n_{k}}^{k}=\bar{c}$, and consider the right-continuous step functions

$$
\zeta_{k}(c)=\sum_{i=1}^{n_{k}-1} \zeta\left(c_{i}^{k}\right) I_{\left[c_{i}^{k}, c_{i+1}^{k}\right)},
$$

such that $\delta\left(\mathcal{S}^{k}\right)=\max _{i=1, \cdots, n_{k}-1}\left(c_{i+1}^{k}-c_{i}^{k}\right) \rightarrow 0$. We have that $\zeta_{k} \rightarrow \zeta$ uniformly, and so both $A^{\zeta_{k}}(c) \rightarrow A^{\zeta}(c)$ and $W^{\zeta_{k}}(x, c) \rightarrow W^{\zeta}(x, c)$ uniformly.

We now look for the maximum of $W^{\zeta}$ among $\zeta \in \mathcal{B}$. We will show that if there exists a function $\zeta_{0} \in \mathcal{B}$ such that

$$
\begin{equation*}
A^{\zeta_{0}}(\underline{c})=\max \left\{A^{\zeta}(\underline{c}): \zeta \in \mathcal{B}\right\} \tag{29}
\end{equation*}
$$

then $W^{\zeta_{0}}(x, c) \geq W^{\zeta}(x, c)$ for all $(x, c) \in[0, \infty) \times[c, \bar{c})$ and $\zeta \in \mathcal{B}$. This follows from 24 and the next lemma, in which we prove that the function $\zeta_{0}$ which maximizes 29) also maximizes $A^{\zeta}(c)$ for any $c \in[\underline{c}, \bar{c})$.

Lemma 5.9 For a given $c \in[\underline{c}, \bar{c})$, define
$\mathcal{B}_{c}=\left\{\zeta\right.$ st. $\zeta:[c, \bar{c}) \rightarrow[0, \infty)$ is Riemann integrable and càdlàg with $\left.\lim _{d \rightarrow \bar{c}^{-}} \zeta(d)<\infty\right\}$.
If $\zeta_{0} \in \mathcal{B}$ satisfies (29), then for any $c \in[\underline{c}, \bar{c})$

$$
A^{\zeta_{0}}(c)=\max \left\{A^{\zeta}(c): \zeta \in \mathcal{B}_{c}\right\} .
$$

Proof. Given $\zeta \in \mathcal{B}$, we can write

$$
A^{\zeta}(\underline{c})=\left(-\int_{\underline{c}}^{c} e^{-\int_{\underline{c}}^{t} b_{1}(\zeta(s), s) d s} b_{0}(\zeta(t), t) d t\right)+\left(e^{-\int_{\underline{c}}^{c} b_{1}(\zeta(s), s) d s}\right) A^{\zeta}(c) .
$$

So

$$
A^{\zeta_{0}}(\underline{c})=\left(-\int_{\underline{c}}^{c} e^{-\int_{\underline{c}}^{t} b_{1}\left(\zeta_{0}(s), s\right) d s} b_{0}\left(\zeta_{0}(t), t\right) d t\right)+\left(e^{-\int_{\underline{c}}^{c} b_{1}\left(\zeta_{0}(s), s\right) d s}\right) \max _{\zeta \in \mathcal{B}_{c}} A^{\zeta}(c) .
$$

Assuming that $\zeta_{0}$ exists, we will use calculus of variations to obtain an implicit equation for $A^{\zeta_{0}}$. First we prove the following technical lemma.

Lemma 5.10 For any $c \in[\underline{c}, \bar{c}]$, we have

$$
\partial_{x} b_{1}(x, c)<0 \text { for } x>0 \text { and } \partial_{x} b_{1}\left(0^{+}, c\right)<0 .
$$

Proof.

$$
\begin{aligned}
\partial_{x} b_{1}(x, c)= & \frac{\theta_{1}^{\prime}(c) e^{2 \theta_{1}(c) x}\left(-1+e^{-\left(\theta_{1}(c)-\theta_{2}(c)\right) x}\left(1+\left(\theta_{1}(c)-\theta_{2}(c)\right) x\right)\right)}{\left(e^{\theta_{1}(c) x}-e^{\theta_{2}(c) x}\right)^{2}} \\
& +\frac{\theta_{2}^{\prime}(c) e^{2 \theta_{2}(c) x}\left(-1+e^{\left(\theta_{1}(c)-\theta_{2}(c)\right) x}\left(1-\left(\theta_{1}(c)-\theta_{2}(c)\right) x\right)\right)}{\left(e^{\theta_{1}(c) x}-e^{\theta_{2}(c) x}\right)^{2}} \\
< & 0
\end{aligned}
$$

and, by Remark 3.1.

$$
\lim _{x \rightarrow 0} \partial_{x} b_{1}(x, c)=-\frac{\theta_{1}^{\prime}(c)+\theta_{2}^{\prime}(c)}{2}<0 .
$$

Let us now find the implicit equation for $A^{\zeta_{0}}$.
Proposition 5.11 If the function $\zeta_{0}$ defined in (29) exists, then $A^{\zeta_{0}}(c)$ satisfies

$$
A^{\zeta_{0}}(c)=-\frac{\partial_{x} b_{0}\left(\zeta_{0}(c), c\right)}{\partial_{x} b_{1}\left(\zeta_{0}(c), c\right)}
$$

for all $c \in[\underline{c}, \bar{c})$. Moreover, $A^{\zeta_{0}}(\bar{c})=0$ and $A^{\zeta_{0}}(c)>0$ for $c \in[\underline{c}, \bar{c})$.

Proof. Consider any function $\zeta_{1} \in \mathcal{B}$ with $\zeta_{1}(\bar{c})=0$ then

$$
A^{\zeta_{0}+\varepsilon \zeta_{1}}(\underline{c})=-\int_{\underline{c}}^{\bar{c}} e^{-\int_{\underline{c}}^{c} b_{1}\left(\zeta_{0}(s)+\varepsilon \zeta_{1}(s), s\right) d s} b_{0}\left(\zeta_{0}(c)+\varepsilon \zeta_{1}(c), c\right) d c .
$$

Taking the derivative with respect to $\varepsilon$ and taking $\varepsilon=0$, we get

$$
\begin{aligned}
0=\left.\partial_{\varepsilon}\left(A^{\zeta_{0}+\varepsilon \zeta_{1}}\right)(\underline{c})\right|_{\varepsilon=0}= & \int_{\underline{c}}^{\bar{c}}\left(\left(\int_{\underline{c}}^{c} \partial_{x} b_{1}\left(\zeta_{0}(s), s\right) \zeta_{1}(s) d s\right) e^{-\int_{\underline{c}}^{c} b_{1}\left(\zeta_{0}(s), s\right) d s} b_{0}\left(\zeta_{0}(c), c\right)\right) d c \\
& -\int_{\underline{c}}^{\bar{c}}\left(e^{-\int_{\underline{c}}^{c} b_{1}\left(\zeta_{0}(s), s\right) d s} \partial_{x} b_{0}\left(\zeta_{0}(c), c\right) \zeta_{1}(c)\right) d c . \\
= & \int_{\underline{c}}^{\bar{c}}\left(\partial_{x} b_{1}\left(\zeta_{0}(c), c\right) \zeta_{1}(c)\left(\int_{c}^{\bar{c}} e^{-\int_{\underline{\underline{c}}}^{u} b_{1}\left(\zeta_{0}(s), s\right) d s} b_{0}\left(\zeta_{0}(u), u\right) d u\right)\right) d c \\
& -\int_{\underline{c}}^{\bar{c}}\left(e^{-\int_{\underline{c}}^{c} b_{1}\left(\zeta_{0}(s), s\right) d s} \partial_{x} b_{0}\left(\zeta_{0}(c), c\right) \zeta_{1}(c)\right) d c .
\end{aligned}
$$

And so,
$0=-\int_{\underline{c}}^{\bar{c}}\left(e^{-\int_{\underline{c}}^{c} b_{1}\left(\zeta_{0}(s), s\right) d s} \partial_{x} b_{0}\left(\zeta_{0}(c), c\right)-\partial_{x} b_{1}\left(\zeta_{0}(c), c\right)\left(\int_{c}^{\bar{c}} e^{-\int_{\underline{x}}^{u} b_{1}\left(\zeta_{0}(s), s\right) d s} b_{0}\left(\zeta_{0}(u), u\right) d u\right)\right) \zeta_{1}(c) d c$.
Since this holds for any $\zeta_{1} \in \mathcal{B}$ with $\zeta_{1}(\bar{c})=0$, we obtain that for any $c \in[c, \bar{c})$

$$
\begin{aligned}
0 & =\partial_{x} b_{0}\left(\zeta_{0}(c), c\right)-\partial_{x} b_{1}\left(\zeta_{0}(c), c\right)\left(\int_{c}^{\bar{c}} e^{-\int_{c}^{u} b_{1}\left(\zeta_{0}(s), s\right) d s} b_{0}\left(\zeta_{0}(u), u\right) d u\right) \\
& =\partial_{x} b_{0}\left(\zeta_{0}(c), c\right)-\partial_{x} b_{1}\left(\zeta_{0}(c), c\right) A^{\zeta_{0}}(c)
\end{aligned}
$$

Using Lemma 5.10, we get the implicit equation for $\zeta_{0}$. By definition $A^{\zeta_{0}}(\bar{c})=0$. Now take $c \in[\underline{c}, \bar{c})$, and the constant step function $\zeta \in \mathcal{B}$ defined as $\zeta \equiv x_{0}$ where $x_{0}$ satisfies

$$
\frac{c}{q}\left(1-e^{\theta_{2}(c) x_{0}}\right)<\frac{\bar{c}}{q}\left(1-e^{\theta_{2}(\bar{c}) x_{0}}\right) .
$$

Then

$$
A^{\zeta_{0}}(c) \geq A^{\zeta}(c)=\frac{\frac{\bar{c}}{q}\left(1-e^{\theta_{2}(\bar{c}) x_{0}}\right)-\frac{c}{q}\left(1-e^{\theta_{2}(c) x_{0}}\right)}{e^{\theta_{1}(c) x_{0}}-e^{\theta_{2}(c) x}}>0
$$

From now on, we extend the definition of $\zeta_{0}$ to $[\underline{c}, \bar{c}]$ as

$$
\zeta_{0}(\bar{c}):=\lim _{d \rightarrow \bar{c}^{-}} \zeta_{0}(d)
$$

Since $A^{\zeta_{0}}(\bar{c})=0$, we get from Proposition 5.11

$$
\begin{equation*}
\partial_{x} b_{0}(\zeta(\bar{c}), \bar{c})=0 \tag{30}
\end{equation*}
$$

and since $A^{\zeta_{0}}(c)>0$ for $c \in[\underline{c}, \bar{c})$, we obtain that

$$
\partial_{x} b_{0}\left(\zeta_{0}(c), c\right)>0
$$

In the next proposition we show that, under some assumptions, the function $\zeta_{0}:[\underline{c}, \bar{c}] \rightarrow[0, \infty)$ is the unique solution of the first order differential equation

$$
\begin{equation*}
\zeta^{\prime}(c)=\left(\frac{-b_{0}\left(\partial_{x} b_{1}\right)^{2}+b_{1} \partial_{x} b_{0} \partial_{x} b_{1}-\partial_{x c} b_{0} \partial_{x} b_{1}+\partial_{x c} b_{1} \partial_{x} b_{0}}{\partial_{x x} b_{0} \partial_{x} b_{1}-\partial_{x x} b_{1} \partial_{x} b_{0}}\right)(\zeta(c), c) \tag{31}
\end{equation*}
$$

with boundary condition 30).

Proposition 5.12 If $\zeta_{0}(c)$ defined in (29) satisfies

$$
\begin{equation*}
\left(\partial_{x x} b_{0} \partial_{x} b_{1}-\partial_{x x} b_{1} \partial_{x} b_{0}\right)\left(\zeta_{0}(c), c\right) \neq 0 \tag{32}
\end{equation*}
$$

then $\zeta_{0}$ is infinitely differentiable and it is the unique solution of (31) with boundary condition (30).

Proof. From 26, we have

$$
\begin{equation*}
\left(-\frac{\partial_{x} b_{0}\left(\zeta_{0}(c), c\right)}{\partial_{x} b_{1}\left(\zeta_{0}(c), c\right)}\right)^{\prime}=b_{0}\left(\zeta_{0}(c), c\right)+b_{1}\left(\zeta_{0}(c), c\right)\left(\frac{\partial_{x} b_{0}\left(\zeta_{0}(c), c\right)}{-\partial_{x} b_{1}\left(\zeta_{0}(c), c\right)}\right) \tag{33}
\end{equation*}
$$

By Assumption (32), the function

$$
G(\zeta, c):=-\frac{\partial_{x} b_{0}(\zeta, c)}{\partial_{x} b_{1}(\zeta, c)}
$$

satisfies

$$
\partial_{\zeta} G\left(\zeta_{0}(c), c\right)=-\left(\frac{\partial_{x x} b_{0}\left(\zeta_{0}(c), c\right) \partial_{x} b_{1}\left(\zeta_{0}(c), c\right)-\partial_{x x} b_{1}\left(\zeta_{0}(c), c\right) \partial_{x} b_{0}\left(\zeta_{0}(c), c\right)}{\partial_{x} b_{1}\left(\zeta_{0}(c), c\right)^{2}}\right) \neq 0
$$

Hence, from (33), we have that $\zeta_{0}(c)$ is differentiable and we get the differential equation (31) for $\zeta_{0}$. We obtain by a recursive argument that $\zeta_{0}(c)$ is infinitely differentiable.

In the next proposition, we state that the value function $W^{\zeta_{0}}$ satisfies a smooth-pasting property on the smooth free-boundary curve

$$
\mathcal{R}\left(\zeta_{0}\right)=\left\{\left(\zeta_{0}(c), c\right) \text { with } c \in[\underline{c}, \bar{c})\right\} .
$$

We also show that, under some conditions, $\zeta_{0}$ is the unique continuous function $\zeta \in \mathcal{B}$ such that the associated $\zeta$-value function $W^{\zeta}$ satisfies the smooth-pasting property at the curve $\mathcal{R}(\zeta)$.

Proposition 5.13 If $\zeta_{0}$ defined in (29) satisfies (32), then $W^{\zeta_{0}}$ satisfies the smooth-pasting property

$$
W_{c x}^{\zeta_{0}}\left(\zeta_{0}(c), c\right)=W_{c c}^{\zeta_{0}}\left(\zeta_{0}(c), c\right)=0 \text { for } c \in[\underline{c}, \bar{c}]
$$

Conversely, let $h:[0, \infty) \times[\underline{c}, \bar{c}] \rightarrow[0, \infty)$ with $h(x, \bar{c})=V^{\{\bar{c}\}}(x, \bar{c})$ and $h(0, c)=0$ for $c \in[\underline{c}, \bar{c})$. Assume that for $c \in[\underline{c}, \bar{c})$,

$$
\zeta(c):=\sup \left\{y: \mathcal{L}^{c}(h)(x, c)=0 \text { for } 0 \leq x \leq y\right\}
$$

is a positive and continuous function in $\mathcal{B}$ satisfying

$$
\partial_{c} h(\zeta(c), c)=\partial_{c x} h(\zeta(c), c)=0
$$

and $\left(\partial_{x x} b_{0} \partial_{x} b_{1}-\partial_{x x} b_{1} \partial_{x} b_{0}\right)(\zeta(c), c) \neq 0$ for $c \in[\underline{c}, \bar{c}) ;$ then $\zeta$ coincides with $\zeta_{0}$ and $h(x, c)=$ $W^{\zeta_{0}}(x, c)$ for $0 \leq x \leq \zeta(c)$ and $c \in[\underline{c}, \bar{c}]$.

Proof. Let us define for $x \geq 0$ and $c \in[\underline{c}, \bar{c}]$ the function

$$
\begin{equation*}
H(x, c):=\frac{c}{q}\left(1-e^{\theta_{2}(c) x}\right)+a(c)\left(e^{\theta_{1}(c) x}-e^{\theta_{2}(c) x}\right), \tag{34}
\end{equation*}
$$

where $a:[\underline{c}, \bar{c}] \rightarrow[0, \infty)$ is a function with $a(\bar{c})=0$. Note that $H$ satisfies $\mathcal{L}^{c}(W(x, c))=0$ for all $x \geq 0, H(0, c)=0$ and $H(x, \bar{c})=\frac{\bar{c}}{q}\left(1-e^{\theta_{2}(c) x}\right)$. We have,

$$
\partial_{x} H(x, c)=-\frac{c}{q} \theta_{2}(c) e^{\theta_{2}(c) x}+a(c)\left(\theta_{1}(c) e^{\theta_{1}(c) x}-\theta_{2}(c) e^{\theta_{2}(c) x}\right)
$$

If $a(c)$ is differentiable,

$$
\begin{equation*}
\partial_{c} H(x, c)=\left(e^{\theta_{1}(c) x}-e^{\theta_{2}(c) x}\right)\left(-b_{0}(x, c)-b_{1}(x, c) a(c)+a^{\prime}(c)\right), \tag{35}
\end{equation*}
$$

and

$$
\begin{aligned}
\partial_{c x} H(x, c)=\partial_{x c} H(x, c)= & \left(\theta_{1}(c) e^{\theta_{1}(c) x}-\theta_{2}(c) e^{\theta_{2}(c) x}\right)\left(-b_{0}(x, c)-b_{1}(x, c) a(c)+a^{\prime}(c)\right) \\
& +\left(e^{\theta_{1}(c) x}-e^{\theta_{2}(c) x}\right)\left(-\partial_{x} b_{0}(x, c)-\partial_{x} b_{1}(x, c) a(c)\right) .
\end{aligned}
$$

In the case that $a(c)=A^{\zeta_{0}}(c)$, take $H=H^{\zeta_{0}}$ as defined in 24), by 26) and Proposition 5.11, we obtain $\partial_{c x} H^{\zeta_{0}}\left(\zeta_{0}(c), c\right)=0$. Since $W^{\zeta_{0}}(x, c)=H^{\zeta_{0}}(x, c)$ for $x<\zeta_{0}(c)$ and $W^{\zeta_{0}}(x, c)=$ $H^{\zeta_{0}}(x, C(x, c))$ for $x \geq \zeta_{0}(c)$, we get $\partial_{c} W^{\zeta_{0}}(x, c)=0$ for $x \geq \zeta_{0}(c)$ and so $\partial_{c x} W^{\zeta_{0}}(\zeta(c), c)=0$. From (35), we get

$$
\partial_{c} H^{\zeta_{0}}(x, c)=\left(e^{\theta_{1}(c) x}-e^{\theta_{2}(c) x}\right)\left(b_{0}\left(\zeta_{0}(c), c\right)-b_{0}(x, c)+\left(b_{1}\left(\zeta_{0}(c), c\right)-b_{1}(x, c)\right) A^{\zeta_{0}}(c)\right)
$$

Hence, from 32, we have that $\partial_{c c} H^{\zeta_{0}}$ exists. Since $\partial_{c} H^{\zeta_{0}}\left(\zeta_{0}(c), c\right)=0$ for $c \in[\underline{c}, \bar{c}]$,

$$
\begin{aligned}
0 & =\frac{d}{d c}\left(\partial_{c} H^{\zeta_{0}}\left(\zeta_{0}(c), c\right)\right) \\
& =\partial_{c c} H^{\zeta_{0}}\left(\zeta_{0}(c), c\right)+\partial_{c x} H^{\zeta_{0}}\left(\zeta_{0}(c), c\right) \zeta_{0}^{\prime}(c) \\
& =\partial_{c c} H^{\zeta_{0}}\left(\zeta_{0}(c), c\right)
\end{aligned}
$$

Finally, since $W^{\zeta_{0}}(x, c)=H^{\zeta_{0}}(x, C(x, c))$ if $x \geq \zeta_{0}(c)$, we get $\partial_{c c} W^{\zeta_{0}}(x, c)=0$ if $x \geq \zeta_{0}(c)$ and so $\partial_{c c} W^{\zeta_{0}}\left(\zeta_{0}(c), c\right)=0$.

Conversely, note that there exists $a(c)$ such that $h(x, c)=H(x, c)$ defined in 34) for $x<\zeta(c)$; the existence of $\partial_{c} h$ implies that $a(c)$ is differentiable. Hence,

$$
0=\partial_{c} h(\zeta(c), c)=\left(e^{\theta_{1}(c) \zeta(c)}-e^{\theta_{2}(c) \zeta(c)}\right)\left(-b_{0}(\zeta(c), c)-b_{1}(\zeta(c), c) a(c)+a^{\prime}(c)\right)
$$

which implies

$$
a^{\prime}(c)=b_{0}(\zeta(c), c)+b_{1}(\zeta(c), c) a(c)
$$

Also,

$$
\begin{aligned}
0=\partial_{c x} h(\zeta(c), c)= & \left(e^{\theta_{1}(c) \zeta(c)}-e^{\theta_{2}(c) \zeta(c)}\right)\left(-\partial_{x} b_{0}(\zeta(c), c)-\partial_{x} b_{1}(\zeta(c), c) a(c)\right) \\
& +\left(\theta_{1}(c) e^{\theta_{1}(c) \zeta(c)}-\theta_{2}(c) e^{\theta_{2}(c) \zeta(c)}\right)\left(-b_{0}(\zeta(c), c)-b_{1}(\zeta(c), c) a(c)+a^{\prime}(c)\right)
\end{aligned}
$$

implies

$$
\partial_{x} b_{0}(\zeta(c), c)=\partial_{x} b_{1}(\zeta(c), c) a(c)
$$

Since $\left(\partial_{x x} b_{0} \partial_{x} b_{1}-\partial_{x x} b_{1} \partial_{x} b_{0}\right)(\zeta(c), c) \neq 0$, both $\zeta$ and $\zeta_{0}$ satisfy the same equation and so they coincide.

In the next proposition, we show more regularity for $W^{\zeta_{0}}$ in the case that $\zeta_{0}$ is increasing.
Proposition 5.14 If $\zeta_{0}$ defined in 29, is increasing and satisfies (32), then $W^{\zeta_{0}}$ is (2,1)differentiable. Also, since the inverse $\zeta_{0}^{-1}$ exists, $C(x, c)$ can be written in a simpler way:

$$
C(x, c)= \begin{cases}\bar{c} & \text { if } \quad \zeta_{0}(\bar{c}) \leq x \\ \zeta_{0}^{-1}(x) & \text { if } \quad \zeta_{0}(c) \leq x<\zeta_{0}(\bar{c})\end{cases}
$$

Proof. It is enough to prove that $\partial_{x x} W^{\zeta_{0}}\left(x^{+}, c\right)=\partial_{x x} W^{\zeta_{0}}\left(x^{-}, c\right)$ for $\zeta_{0}(c) \leq x<\zeta_{0}(\bar{c})$. We have

$$
\begin{aligned}
\partial_{x} W^{\zeta_{0}}\left(x^{+}, c\right) & =\partial_{x} H^{\zeta_{0}}\left(x, \zeta_{0}^{-1}(x)\right)+\partial_{c} H^{\zeta_{0}}\left(x, \zeta_{0}^{-1}(x)\right)\left(\zeta_{0}^{-1}\right)^{\prime}(x) \\
& =\partial_{x} H^{\zeta_{0}}\left(x, \zeta_{0}^{-1}(x)\right) \\
& =\partial_{x} W^{\zeta_{0}}\left(x^{-}, c\right)
\end{aligned}
$$

And so,

$$
\begin{aligned}
\partial_{x x} W^{\zeta_{0}}\left(x^{+}, c\right) & =\partial_{x x} H^{\zeta_{0}}\left(x, \zeta_{0}^{-1}(x)\right)+\partial_{c x} H^{\zeta_{0}}\left(x, \zeta_{0}^{-1}(x)\right)\left(\zeta_{0}^{-1}\right)^{\prime}(x) \\
& =\partial_{x x} H^{\zeta_{0}}\left(x, \zeta_{0}^{-1}(x)\right) \\
& =\partial_{x x} W^{\zeta_{0}}\left(x^{-}, c\right)
\end{aligned}
$$

### 5.3 Optimal strategies for the closed interval $S=[\underline{c}, \bar{c}]$

First in this section, we give a verification result in order to check if a $\zeta$-value function $W^{\zeta}$ is the optimal value function $V$. Our conjecture is that the solution $\bar{\zeta}$ of (31) with boundary condition exists and is non-decreasing in $[\underline{c}, \bar{c}]$, and that $W^{\bar{\zeta}}$ coincides with $V$, so that there exists an optimal curve strategy.

Using Proposition 5.1, we know that the conjecture holds for $\bar{c} \leq q \sigma^{2} /(2 \mu)$ taking $\zeta_{0} \equiv 0$. In the case that $\bar{c}>q \sigma^{2} /(2 \mu)$, we will show that $\bar{\zeta}$ exists and is increasing and $W^{\bar{\zeta}}=V$ for $[\bar{c}-\varepsilon, \bar{c}]$ for $\varepsilon>0$ small enough. We were not able to prove the conjecture in the general case, although it holds in our numerical explorations (see Section 6). However, we will prove that the $\zeta$-value functions $W^{\zeta}$ are $\varepsilon$-optimal in the following sense: There exists a sequence $\zeta_{n} \in \mathcal{B}$ such that $W^{\zeta_{n}}$ converges uniformly to the optimal value function $V$.

We state now a verification result for checking whether the $\zeta$-value function $W^{\zeta}$ with $\zeta$ continuous is the optimal value function $V$. In this verification result it is not necessary to use viscosity solutions because the proposed value function solves the HJB equation in a classical way. We will check these verification conditions for the limit value function associated to the unique solution of (31) with boundary condition (30) (if exists).

Proposition 5.15 If there exists a smooth function $\bar{\zeta}$ such that the $\bar{\zeta}$-value function $W^{\bar{\zeta}}$ is (2,1)-differentiable and satisfies

$$
\partial_{x} W^{\bar{\zeta}}(\bar{\zeta}(c), c) \leq 1 \text { for } c \in[\underline{c}, \bar{c}] \quad \text { and } \quad \partial_{c} W^{\bar{\zeta}}(x, c) \leq 0 \text { for } x \in[0, \bar{\zeta}(c)) \text { and } c \in[\underline{c}, \bar{c}) \text {, }
$$

then $W^{\bar{\zeta}}=V$.
Proof. We have that $\partial_{x} W^{\bar{\zeta}}(x, \bar{c}) \leq 1$ for $x \geq \bar{\zeta}(\bar{c})$ because $W^{\bar{\zeta}}(\cdot, \bar{c})$ is concave and $\partial_{x} W^{\bar{\zeta}}(\bar{\zeta}(\bar{c}), \bar{c}) \leq$ 1. Since $\mathcal{L}^{c} W^{\bar{\zeta}}(x, c)=0$ for $x \in[0, \bar{\zeta}(c)), c \in[\underline{c}, \bar{c}) ; \partial_{c} W^{\bar{\zeta}}(x, c)=0$ for $\left.x \geq \bar{\zeta}(c)\right), c \in[\underline{c}, \bar{c})$ and $W^{\bar{\zeta}}(\cdot, \bar{c})=V(\cdot, \bar{c})$; by Theorem 3.3 it is sufficient to prove that $\mathcal{L}^{c} W^{\bar{\zeta}}(x, c) \leq 0$ for $\left.x \geq \bar{\zeta}(c)\right)$, $c \in[\underline{c}, \bar{c})$. In this case, we have that

$$
C(x, c)=\max \{h \in[c, \bar{c}]: \bar{\zeta}(d) \leq x \text { for } d \in[c, h)\}
$$

satisfies $C(x, c) \geq c$, and also either $C(x, c)=\bar{c}$ or $\bar{\zeta}(C(x, c))=x$. So, we obtain $\mathcal{L}^{C(x, c)} V(x, C(x, c))=$ 0 and then

$$
\begin{aligned}
\mathcal{L}^{c} V(x, c) & =\mathcal{L}^{C(x, c)} V(x, c)+(C(x, c)-c)\left(\partial_{x} V(x, C(x, c))-1\right) \\
& =(C(x, c)-c)\left(V_{x}(x, C(x, c))-1\right) \leq 0 .
\end{aligned}
$$

Next we see that there exists a unique solution $\bar{\zeta}$ of 31 with boundary condition at least in $[\bar{c}-\varepsilon, \bar{c}]$ for some $\varepsilon>0$. First, let us study the boundary condition (30) in the case $\bar{c}$ $>q \sigma^{2} /(2 \mu)$.

Lemma 5.16 If $\bar{c}>q \sigma^{2} /(2 \mu)$, there exists a unique $\bar{z}>0$ such that $\partial_{x} b_{0}(\bar{z}, \bar{c})=0$; moreover $\partial_{x} b_{0}(x, \bar{c})<0$ for $x \in[0, \bar{z})$ and $\partial_{x} b_{0}(x, \bar{c})>0$ for $x \in(\bar{z}, \infty)$. Also $\partial_{x x} b_{0}(\bar{z}, \bar{c})>0$.

Proof. For $x>0$,

$$
\begin{equation*}
\partial_{x} b_{0}(x, c)=\frac{e^{\theta_{1}(c) x} \theta_{1}(c)-e^{\theta_{2}(c) x} \theta_{2}(c)-c e^{2 \theta_{2}(c) x} \theta_{2}^{\prime}(c)+e^{\left(\theta_{1}(c)+\theta_{2}(c)\right) x}\left(-\theta_{1}(c)+\theta_{2}(c)+c \theta_{2}^{\prime}(c)\left(1-x \theta_{1}(c)+x \theta_{2}(c)\right)\right.}{q\left(e^{\theta_{1}(c) x}-e^{\theta_{2}(c) x}\right)^{2}} \tag{36a}
\end{equation*}
$$

and so

$$
\lim _{x \rightarrow 0^{+}} \partial_{x} b_{0}(x, \bar{c})=-\frac{1}{2 q}\left(\frac{\theta_{1}(\bar{c}) \theta_{2}(\bar{c})}{\theta_{1}(\bar{c})-\theta_{2}(\bar{c})}+\bar{c} \theta_{2}^{\prime}(\bar{c})\right)=\frac{\bar{c}^{2}+q \sigma^{2}-\bar{c}\left(\mu+\sqrt{(\bar{c}-\mu)^{2}+2 q \sigma^{2}}\right)}{2 q \sigma^{2} \sqrt{(\bar{c}-\mu)^{2}+2 q \sigma^{2}}} .
$$

Hence, $\lim _{x \rightarrow 0^{+}} \partial_{x} b_{0}(x, \bar{c}) \geq 0$ for $\bar{c} \leq q \sigma^{2} /(2 \mu)$ and $\lim _{x \rightarrow 0^{+}} \partial_{x} b_{0}(x, \bar{c})<0$ for $\bar{c}>q \sigma^{2} /(2 \mu)$. Also

$$
\lim _{x \rightarrow \infty} \partial_{x} b_{0}(x, \bar{c}) e^{\theta_{1}(\bar{c}) x}=\frac{\theta_{1}(\bar{c})}{q}>0 .
$$

So, for $\bar{c}>q \sigma^{2} /(2 \mu)$ there exists (at least one) $\bar{z}>0$ such that $\partial_{x} b_{0}(\bar{z}, \bar{c})=0$.
We are showing next that $\partial_{x} b_{0}(x, \bar{c})=0$ for $x>0$ implies that $\partial_{x x} b_{0}(x, \bar{c})>0$. Consequently the result follows.

From 36a, we can write

$$
\partial_{x} b_{0}(x, \bar{c}) q\left(e^{\theta_{1}(\bar{c}) x}-e^{\theta_{2}(\bar{c}) x}\right)^{2}=g_{11}(x, \bar{c}) \theta_{2}^{\prime}(\bar{c})+g_{10}(x, \bar{c})
$$

and

$$
\partial_{x x} b_{0}(x, \bar{c}) q\left(e^{\theta_{1}(\bar{c}) x}-e^{\theta_{2}(\bar{c}) x}\right)^{3}=g_{21}(x, \bar{c}) \theta_{2}^{\prime}(\bar{c})+g_{20}(x, \bar{c})
$$

where

$$
\begin{gathered}
g_{11}(x, \bar{c})=-\bar{c} e^{2 \theta_{2}(\bar{c}) x}\left(1-e^{\left(\theta_{1}(\bar{c})-\theta_{2}(\bar{c})\right) x}\left(1-x\left(\theta_{1}(\bar{c})-\theta_{2}(\bar{c})\right)\right)\right. \\
g_{10}(x, \bar{c})=-\theta_{1}(\bar{c}) e^{\theta_{1}(\bar{c}) x}\left(e^{\theta_{2}(\bar{c}) x}-1\right)+\theta_{2}(\bar{c}) e^{\theta_{2}(\bar{c}) x}\left(e^{\theta_{1}(\bar{c}) x}-1\right) \\
g_{21}(x, \bar{c})=\bar{c} e^{\left(\theta_{1}(\bar{c})+\theta_{2}(\bar{c})\right) x}\left(\theta_{1}(\bar{c})-\theta_{2}(\bar{c})\right)\left(e^{\theta_{1}(\bar{c}) x}\left(-2+x \theta_{1}(\bar{c})-x \theta_{2}(\bar{c})\right)+e^{\theta_{2}(\bar{c}) x}\left(2+x \theta_{1}(\bar{c})-x \theta_{2}(\bar{c})\right)\right) \\
g_{20}(x, \bar{c})=e^{\theta_{1}(\bar{c}) x} \theta_{1}^{2}(\bar{c})\left(e^{\theta_{2}(\bar{c}) x}-1\right)\left(e^{\theta_{1}(\bar{c}) x}+e^{\theta_{2}(\bar{c}) x}\right)-2 e^{\left(\theta_{1}(\bar{c})+\theta_{2}(\bar{c})\right) x} \theta_{1}(\bar{c}) \theta_{2}(\bar{c})\left(-2+e^{\theta_{1}(\bar{c}) x}+e^{\theta_{2}(\bar{c}) x}\right) \\
+e^{\theta_{2}(\bar{c}) x} \theta_{2}^{2}(\bar{c})\left(e^{\theta_{1}(\bar{c}) x}-1\right)\left(e^{\theta_{1}(\bar{c}) x}+e^{\theta_{2}(\bar{c}) x}\right)
\end{gathered}
$$

If $x>0$, take $u=x\left(\theta_{1}(\bar{c})-\theta_{2}(\bar{c})\right)>0$, we can write

$$
-\frac{g_{11}(x, \bar{c})}{\bar{c} e^{2 \theta_{2}(\bar{c}) x}}=1-e^{u}(1-u)>0
$$

which implies that $g_{11}(x, \bar{c})<0$.
Consider now

$$
\begin{aligned}
g(x, \bar{c}) & :=\partial_{x x} b_{0}(x, \bar{c}) q\left(e^{\theta_{1}(\bar{c}) x}-e^{\theta_{2}(\bar{c}) x}\right)^{3} g_{11}(x, \bar{c})-\partial_{x} b_{0}(x, \bar{c}) q\left(e^{\theta_{1}(\bar{c}) x}-e^{\theta_{2}(\bar{c}) x}\right)^{2} g_{21}(x, \bar{c}) \\
& =g_{20}(x, \bar{c}) g_{11}(x, \bar{c})-g_{10}(x, \bar{c}) g_{21}(x, \bar{c}) .
\end{aligned}
$$

We are going to prove that $g(x, \bar{c})<0$ for $x>0$. For that purpose, take

$$
g_{0}(x, \bar{c}):=-\frac{g(x, \bar{c})}{\bar{c}\left(e^{\theta_{1}(\bar{c}) x}-e^{\theta_{2}(\bar{c}) x}\right)^{2} e^{2 \theta_{2}(\bar{c}) x} \theta_{1}^{2}(\bar{c})} .
$$

Calling $t=-\frac{\theta_{2}(\bar{c})}{\theta_{1}(\bar{c})}>0$ and $s=\theta_{1}(\bar{c}) x>0$, and we can write

$$
g_{0}(x, \bar{c})=-t^{2}+e^{s}(1+t)^{2}+e^{s+s t}\left(-1-2 t+s t+s t^{2}\right)
$$

Then $g_{0}(x, \bar{c})>0$ for $x>0$, because $g_{0}(s, 0)=\partial_{t} g_{0}(s, 0)=0, \partial_{t}^{2} g_{0}(s, 0)=-2+\left(2-2 s+s^{2}\right) e^{s}>$ 0 and

$$
\partial_{t}^{3} g_{0}(s, t)=s^{3} e^{s+s t}\left(2+4 t+s t+s t^{2}\right)>0
$$

for $s, t>0$. Finally, if $\partial_{x} b_{0}(\bar{z}, \bar{c})=0$ for $\bar{z}>0$, since

$$
\begin{aligned}
g(\bar{z}, \bar{c}) & =\partial_{x x} b_{0}(\bar{z}, \bar{c}) q\left(e^{\theta_{1}(\bar{c}) \bar{z}}-e^{\theta_{2}(\bar{c}) \bar{z}}\right)^{3} g_{11}(\bar{z}, \bar{c})-\partial_{x} b_{0}(\bar{z}, \bar{c}) q\left(e^{\theta_{1}(\bar{c}) \bar{z}}-e^{\theta_{2}(\bar{c}) \bar{z}}\right)^{2} g_{21}(\bar{z}, \bar{c}) \\
& =\partial_{x x} b_{0}(\bar{z}, \bar{c}) q\left(e^{\theta_{1}(\bar{c}) \bar{z}}-e^{\theta_{2}(\bar{c}) \bar{z}}\right)^{3} g_{11}(\bar{z}, \bar{c}),
\end{aligned}
$$

$g(\bar{z}, \bar{c})<0$ and $g_{11}(\bar{z}, \bar{c})<0$, we get that $\partial_{x x} b_{0}(\bar{z}, \bar{c})>0$.
Proposition 5.17 In the case $\bar{c}>q \sigma^{2} /(2 \mu)$ there exists a unique increasing solution $\bar{\zeta}$ of (31) with boundary condition (30) in $[\bar{c}-\varepsilon, \bar{c}]$ for some $\varepsilon>0$.

Proof. From Lemma 5.16, $\bar{\zeta}(\bar{c})=\bar{z}$. By (31) and since the functions $b_{0}$ and $b_{1}$ are infinitely differentiable, it suffices to prove that

$$
\left(\partial_{x x} b_{0} \partial_{x} b_{1}-\partial_{x x} b_{1} \partial_{x} b_{0}\right)(x, c) \neq 0
$$

in a neighborhood of $(\bar{z}, \bar{c})$. From Lemmas 5.10 and 5.16

$$
\left(\partial_{x x} b_{0} \partial_{x} b_{1}-\partial_{x x} b_{1} \partial_{x} b_{0}\right)(\bar{z}, \bar{c})=\left(\partial_{x x} b_{0} \partial_{x} b_{1}\right)(\bar{z}, \bar{c})<0
$$

The existence of $\bar{\zeta}$ follows by continuity.
In order to show that $\bar{\zeta}$ is increasing in $[\bar{c}-\varepsilon, \bar{c}]$ for some $\varepsilon>0$, it is sufficient to prove

$$
\left(-b_{0}\left(\partial_{x} b_{1}\right)^{2}+b_{1} \partial_{x} b_{0} \partial_{x} b_{1}-\partial_{x c} b_{0} \partial_{x} b_{1}+\partial_{x c} b_{1} \partial_{x} b_{0}\right)(x, c)<0
$$

in a neighborhood of $(\bar{z}, \bar{c})$. Since $\partial_{x} b_{0}(\bar{z}, \bar{c})=0$, we get

$$
\begin{aligned}
& \left(-b_{0}\left(\partial_{x} b_{1}\right)^{2}+b_{1} \partial_{x} b_{0} \partial_{x} b_{1}-\partial_{x c} b_{0} \partial_{x} b_{1}+\partial_{x c} b_{1} \partial_{x} b_{0}\right)(\bar{z}, \bar{c}) \\
& =-\partial_{x} b_{1}(\bar{z}, \bar{c})\left(b_{0}(\bar{z}, \bar{c}) \partial_{x} b_{1}(\bar{z}, \bar{c})+\partial_{x c} b_{0}(\bar{z}, \bar{c})\right) ;
\end{aligned}
$$

and since $\partial_{x} b_{1}(\bar{z}, \bar{c})<0$, it is enough to show that

$$
b_{0}(\bar{z}, \bar{c}) \partial_{x} b_{1}(\bar{z}, \bar{c})+\partial_{x c} b_{0}(\bar{z}, \bar{c})<0
$$

Taking $t=-\frac{\theta_{2}(\bar{c})}{\theta_{1}(\bar{c})}>0$ and $u=\frac{-q}{\theta_{2}(\bar{c}) \sigma^{2}} \bar{z}>0$, we can write

$$
\frac{b_{0}(\bar{z}, \bar{c}) \partial_{x} b_{1}(\bar{z}, \bar{c})+\partial_{x c} b_{0}(\bar{z}, \bar{c})}{g_{0}(\bar{z}, \bar{c})}=g_{1}(u, t)+\frac{\overline{c z}}{\sigma^{2}} g_{2}(u, t)
$$

and

$$
\frac{\partial_{x} b_{0}(\bar{z}, \bar{c})}{f_{0}(\bar{z}, \bar{c})}=f_{1}(u, t)+\frac{\overline{c z}}{\sigma^{2}} f_{2}(u, t)=0 ;
$$

where

$$
g_{0}(\bar{z}, \bar{c})=\frac{\bar{z}\left(e^{\theta_{1}(\bar{c}) \bar{z}}-e^{\theta_{2}(\bar{c}) \bar{z}}\right)^{3} q^{2}\left(\theta_{1}(\bar{c})-\theta_{2}(\bar{c})\right)^{3}}{2 \theta_{1}^{3}(\bar{c})\left(-\theta_{2}(\bar{c})\right)}>0
$$

and

$$
f_{0}(\bar{z}, \bar{c})=\frac{\bar{z}\left(e^{\theta_{1}(\bar{c}) \bar{z}}-e^{\theta_{2}(\bar{c}) \bar{z}}\right)^{2} q\left(\theta_{1}(\bar{c})-\theta_{2}(\bar{c})\right)}{\left.2 \theta_{1}(\bar{c})\right)}>0 .
$$

We are going to show that $f_{2}(u, t)<0$ and also

$$
d_{0}(u, t):=g_{1}(u, t) f_{2}(u, t)-f_{1}(u, t) g_{2}(u, t)>0
$$

From these inequalities, we conclude that

$$
\begin{aligned}
b_{0}(\bar{z}, \bar{c}) \partial_{x} b_{1}(\bar{z}, \bar{c})+\partial_{x c} b_{0}(\bar{z}, \bar{c}) & =\frac{g_{0}(\bar{z}, \bar{c})}{f_{2}(u, t)}\left(d_{0}(u, t)+\frac{\partial_{x} b_{0}(\bar{z}, \bar{c})}{f_{0}(\bar{z}, \bar{c})} g_{2}(u, t)\right) \\
& =\frac{g_{0}(\bar{z}, \bar{c})}{f_{2}(u, t)} d_{0}(u, t) \\
& <0 .
\end{aligned}
$$

Let us see first that

$$
f_{2}(u, t)=-t e^{-4 u t}\left(1+(2 u+2 u t-1) e^{2 u+2 u t}\right)<0 .
$$

This holds immediately taking $y=2 u+2 u t$, because $1+(y-1) e^{y}>0$ for $y>0$. Let us see now that $d_{0}(u, t)>0$, we obtain

$$
d_{0}(u, t)=h_{0}(u, t)\left(P_{0}(u, t)+P_{1}(u, t) e^{2 u}+P_{2}(u, t) e^{2 u+2 u t}+P_{3}(u, t) e^{4 u+2 u t}+P_{4}(u, t) e^{4 u+4 u t}\right)
$$

where

$$
\begin{gathered}
h_{0}(u, t)=2 u t(1+t) e^{-8 u t}\left(e^{2 u+2 u t}-1\right)>0 \text { for } t, u>0, \\
P_{0}(u, t)=t^{2}, P_{1}(u, t)=(1+t)(2-t+2 u+2 u t), \\
P_{2}(u, t)=-2-2 u-t+4 u^{2} t-t^{2}+4 u t^{2}+6 u^{2} t^{2}+2 u t^{3}-2 u^{2} t^{4}, \\
P_{3}(u, t)=-2+2 u-t+2 u t+2 u^{2} t+t^{2}-2 u t^{2}+6 u^{2} t^{2}-2 u t^{3}+6 u^{2} t^{3}+2 u^{2} t^{4}
\end{gathered}
$$

and

$$
P_{4}(u, t)=2-2 u+t-6 u t+2 u^{2} t-4 u t^{2}+4 u^{2} t^{2}+2 u^{2} t^{3} .
$$

Defining iteratively

$$
\begin{array}{rlrl}
d_{1}(u, t) & :=\frac{\partial_{u} d_{0}(u, t)}{2(1+t) e^{2 u}}, & d_{2}(u, t) & :=\frac{\partial_{u}^{2} d_{1}(u, t)}{4 e^{2 u t}}, \\
d_{3}(u, t) & :=\frac{\partial_{u}^{3} d_{2}(u, t)}{8(1+t)^{4} e^{2 u}}, & d_{4}(u, t):=\frac{\partial_{u}^{3} d_{3}(u, t)}{e^{2 u t}}
\end{array}
$$

we obtain

$$
\begin{gather*}
d_{1}(0, t)=\partial_{u} d_{1}(0, t)=0,  \tag{37}\\
d_{2}(0, t)=\partial_{u} d_{2}(0, t)=\partial_{u}^{2} d_{2}(0, t)=0,  \tag{38}\\
d_{3}(0, t)=5 t^{2}, \partial_{u} d_{3}(0, t)=2 t^{2}(7+22 t), \partial_{u}^{2} d_{3}(0, t)=8 t^{2}\left(1+18 t+25 t^{2}\right), \tag{39}
\end{gather*}
$$

and
$d_{4}(u, t)=16 t^{3}\left(10+4 u+42 t+29 u t+2 u^{2} t+43 t^{2}+64 u t^{2}+10 u^{2} t^{2}+44 u t^{3}+16 u^{2} t^{3}+8 u^{2} t^{4}\right)$.
Since $d_{4}(u, t)>0$ and the expressions in 39) are positive, we have $d_{3}(u, t)>0$. Similarly, by (38) and (37) we get that $d_{2}(u, t), d_{1}(u, t)$ and finally $d_{0}(u, t)$ are all positive.

In the following proposition, we show that the conjecture holds for $S=[\bar{c}-\varepsilon, \bar{c}]$ with $\varepsilon>0$ small enough.

Proposition 5.18 In the case $\bar{c}>q \sigma^{2} /(2 \mu)$, there exists $\varepsilon>0$ such that $W^{\bar{\zeta}}=V$ in $[0, \infty) \times$ $[\bar{c}-\varepsilon, \bar{c}]$, where $\bar{\zeta}$ is the unique solution of (31) with boundary condition (30).

Proof. Take $\varepsilon>0$ small enough. By Proposition 5.17, $\bar{\zeta}$ is increasing and so, by Proposition 5.14. $W^{\zeta}$ is $(2,1)$-differentiable in $[0, \infty) \times[\bar{c}-\varepsilon, \bar{c}]$. Hence, by Proposition 5.15, we need to prove that $\partial_{x} W^{\bar{\zeta}}(\bar{\zeta}(c), c) \leq 1$ for $c \in[\bar{c}-\varepsilon, \bar{c}]$ and $\partial_{c} W^{\bar{\zeta}}(x, c) \leq 0$ for $(x, c)$ with $c \in[\bar{c}-\varepsilon, \bar{c}]$ and $0 \leq x \leq \bar{\zeta}(c)$.

In order to show that $\partial_{x} W^{\bar{\zeta}}(\bar{\zeta}(c), c) \leq 1$ for $c \in[\bar{c}-\varepsilon, \bar{c}]$, we will see that $\partial_{x} W^{\bar{\zeta}}(\bar{z}, \bar{c})<1$ and the result will follow by continuity. For the unique point

$$
x_{0}:=\frac{1}{\theta_{2}(\bar{c})} \log \left(\frac{-q}{\bar{c} \theta_{2}(\bar{c})}\right)
$$

where $\partial_{x} W^{\bar{\zeta}}\left(x_{0}, \bar{c}\right)=1$, we have that $x_{0}>0$ because $\bar{c}>q \sigma^{2} /(2 \mu)$.

From Lemma 5.16, the condition $\partial_{x} b_{0}\left(x_{0}, \bar{c}\right)<0$ implies $\partial_{x} W^{\bar{\zeta}}(\bar{z}, \bar{c})<1$. Taking $t=$ $\frac{-q}{\theta_{2}(\bar{c}) \bar{c}}>0$ and $r=-\frac{\theta_{1}(\bar{c})}{\theta_{2}(\bar{c})}>0$, we can write

$$
\partial_{x} b_{0}\left(x_{0}, \bar{c}\right)=f\left(x_{0}, \bar{c}\right) g(t, r),
$$

where

$$
f\left(x_{0}, \bar{c}\right)=q^{3} \bar{c} \theta_{2}^{\prime}(\bar{c}) \theta_{2}^{2}(\bar{c})\left(e^{\theta_{1}(\bar{c}) x_{0}}-e^{\theta_{2}(\bar{c}) x_{0}}\right)^{2}>0
$$

and

$$
g(t, r)=-1+\frac{1+(r+1) \log (t)}{t^{r+1}}+\frac{r+1}{r}\left(1-\frac{1+(t+1)(r+1)}{t^{r+1}}\right) .
$$

Since $\bar{c}>q \sigma^{2} /(2 \mu)$, we have that $t<1$. We also get $g(t, r)<0$, because $g(1, r)=0$ and

$$
\partial_{t} g(t, r)=\frac{(r+1)^{2}}{t^{r}}(t+1-\log (t))<0 .
$$

So $\partial_{x} b_{0}\left(x_{0}, \bar{c}\right)<0$.
Let us show now that $\partial_{c} W^{\bar{\zeta}}(x, c) \leq 0$ for $c \in[\bar{c}-\varepsilon, \bar{c}]$ and $0 \leq x \leq \bar{\zeta}(c)$. Since

$$
\partial_{c} W^{\bar{\zeta}}(x, c)=\partial_{c} H^{\bar{\zeta}}(x, c)=\left(e^{\theta_{1}(c) x}-e^{\theta_{2}(c) x}\right)\left(-b_{0}(x, c)-b_{1}(x, c) A^{\bar{\zeta}}(c)+\left(A^{\bar{\zeta}}\right)^{\prime}(c)\right),
$$

we should analyze the sign of

$$
B(x, c):=-b_{0}(x, c)-b_{1}(x, c) A^{\bar{\zeta}}(c)+\left(A^{\bar{\zeta}}\right)^{\prime}(c)
$$

for $c \in[\bar{c}-\varepsilon, \bar{c}]$ and $0 \leq x \leq \bar{\zeta}(c)$. We have that $B(\bar{\zeta}(c), c)=\partial_{x} B(\bar{\zeta}(c), c)=0$. Also, from Lemma 5.16, $\partial_{x x} B(\bar{z}, \bar{c})=-\partial_{x x} b_{0}(\bar{z}, \bar{c})>0$. So, $\partial_{x x} B(x, c)>0$ in some neighborhood

$$
U=\left(\bar{z}-\varepsilon_{1}, \bar{z}+\varepsilon_{1}\right) \times\left(\bar{c}-\varepsilon_{1}, \bar{c}\right] \subset[0, \infty) \times[\bar{c}-\varepsilon, \bar{c}]
$$

of $(\bar{z}, \bar{c})$; and this implies that for any $c \in[\bar{c}-\varepsilon, \bar{c}]$, the function $B(\cdot, c)$ reaches a strict local maximum at $x=\bar{\zeta}(c)$. In particular, by Lemma 5.16. $B(\cdot, \bar{c})=-b_{0}(\cdot, \bar{c})+\left(A^{\bar{\zeta}}\right)^{\prime}(\bar{c})$ reaches the strict global maximum at $x=\bar{z}$ because $\partial_{x} B(\cdot, \bar{c})=-\partial_{x} b_{0}(\cdot, \bar{c})$ changes from positive to negative at this point. This implies that there exists a $\delta>0$ such that $B(x, \bar{c})<-\delta$ for $0 \leq x \leq \bar{z}-\varepsilon_{1}$. Therefore, by continuity arguments, we get $B(x, c)<0$ for $(x, c) \in\left[0, \bar{z}-\varepsilon_{1}\right] \times[\bar{c}-\varepsilon, \bar{c}]$ for some $\varepsilon>0$ small enough and so we conclude the result.

In the next proposition we show that the optimal value function $V$ is a uniform limit of $\zeta$-value functions. Moreover, it is a limit of value functions of extended threshold strategies. The proof uses the convergence result obtained in Section 4

Proposition 5.19 Consider, as in Section 4, a sequence of sets $\mathcal{S}^{n}$ (with $k_{n}$ elements) of the form

$$
\mathcal{S}^{n}=\left\{c_{1}^{n}=\underline{c}<c_{2}^{n}<\cdots<c_{k_{n}}^{n}=\bar{c}\right\}
$$

satisfying $\mathcal{S}^{0}=\{\underline{c}, \bar{c}\}, \mathcal{S}^{n} \subset \mathcal{S}^{n+1}$ and mesh-size $\delta\left(\mathcal{S}^{n}\right):=\max _{i=2, k_{n}}\left(c_{i}^{n}-c_{i-1}^{n}\right) \searrow 0$ as n goes to infinity, and the optimal threshold functions $z_{n}^{*}: \widetilde{S}_{n} \rightarrow[0, \infty)$ defined in Section 5.1. Then, taking $\zeta_{n}(c):=\sum_{i=1}^{k_{n}-1} z_{n}^{*}\left(c_{i}^{n}\right) I_{\left[c_{i}^{n}, c_{i+1}^{n}\right)}$, the $\zeta_{n}$-value functions $W^{\zeta_{n}}$ converge uniformly to the optimal value function $V$.

Proof. Take the functions $V^{n}:[0, \infty) \times[\underline{c}, \bar{c}] \rightarrow \mathbb{R}$ defined in (12). By Proposition 4.1 and Theorem 4.2. $V^{n}$ converges uniformly to the optimal value function $V$. Since by Proposition 2.3 $V$ is Lipschitz with constant $K$ and by definition $V^{n}\left(\cdot, c_{i}^{n}\right)=W^{\zeta_{n}}\left(\cdot, c_{i}^{n}\right)$ for $c_{i}^{n} \in \mathcal{S}^{n}$, we get for $c \in\left[c_{i}^{n}, c_{i+1}^{n}\right)$

$$
\begin{aligned}
0 & \leq V(x, c)-W^{\zeta_{n}}(x, c) \\
& \leq\left(V(x, c)-V\left(x, c_{i}^{n}\right)\right)+\left(V\left(x, c_{i}^{n}\right)-V^{n}\left(x, c_{i}^{n}\right)\right)+\left|W^{\zeta_{n}}\left(x, c_{i}^{n}\right)-W^{\zeta_{n}}(x, c)\right| \\
& \leq K \delta\left(\mathcal{S}^{n}\right)+\max \left|V-V^{n}\right|+\left|W^{\zeta_{n}}\left(x, c_{i}^{n}\right)-W^{\zeta_{n}}(x, c)\right|
\end{aligned}
$$

Hence, in order to prove the result it suffices to show that there exists a $K_{1}>0$ such that

$$
\begin{equation*}
\left|W^{\zeta_{n}}\left(x, c_{i}^{n}\right)-W^{\zeta_{n}}(x, c)\right| \leq K_{1}\left|c_{i}^{n}-c\right| \leq K_{1} \delta\left(\mathcal{S}^{n}\right) \tag{40}
\end{equation*}
$$

We have that $W^{\zeta_{n}}\left(x, c_{i+1}^{n}\right)=W^{\zeta_{n}}(x, c)$ for $x \geq \zeta_{n}\left(c_{i}^{n}\right)$. So if $\zeta_{n}\left(c_{i}^{n}\right)>0$ it remains to prove (40) for $0<x<\zeta_{n}\left(c_{i}^{n}\right)$. Let us define

$$
h(x, c):=\frac{e^{\theta_{1}(c) x}-e^{\theta_{2}(c) x}}{e^{\theta_{1}(c) \zeta_{n}\left(c_{i}\right)}-e^{\theta_{2}(c) \zeta_{n}\left(c_{i}\right)}}
$$

and

$$
u(x, c):=\left(H^{\zeta_{n}}\left(\zeta_{n}\left(c_{i}\right), c_{i+1}\right)-\frac{c}{q}\left(1-e^{\theta_{2}(c) \zeta_{n}\left(c_{i}\right)}\right)\right) h(x, c)
$$

We can write

$$
\begin{aligned}
\left|W^{\zeta_{n}}\left(x, c_{i}^{n}\right)-W^{\zeta_{n}}(x, c)\right| & =\left|\begin{array}{l}
H^{\zeta_{n}}\left(x, c_{i}^{n}\right)-H^{\zeta_{n}}(x, c) \mid \\
\end{array} \leq\left|\frac{c_{i}^{n}}{q}\left(1-e^{\theta_{2}\left(c_{i}^{n}\right) x}\right)-\frac{c}{q}\left(1-e^{\theta_{2}(c) x}\right)\right|+\left|u(x, c)-u\left(x, c_{i}\right)\right|\right.
\end{aligned}
$$

It is straightforward to see that there exists $K_{1}^{1}$ such that

$$
\left|\frac{c_{i}^{n}}{q}\left(1-e^{\theta_{2}\left(c_{i}^{n}\right) x}\right)-\frac{c}{q}\left(1-e^{\theta_{2}(c) x}\right)\right| \leq K_{1}^{1}\left|c-c_{i}^{n}\right| \text { for some } K_{1}^{1}>0
$$

Since $h(\cdot, c)$ is increasing, we obtain $h(c, 0)=0<h(x, c) \leq h\left(\zeta_{n}\left(c_{i}\right), c\right)=1$. Also, we have that $0 \leq H^{\zeta_{n}}(x, c) \leq V(x, c) \leq \bar{c} / q$ for $0<x<\zeta_{n}\left(c_{i}^{n}\right)$; so using that $\theta_{2}(c)<0$ it is easy to show that there exist constants $K_{1}^{2}, K_{1}^{3}>0$ such that

$$
\begin{aligned}
\left|\partial_{c} u(x, c)\right| \leq & \left|-\frac{1}{q}\left(1-e^{\theta_{2}(c) \zeta_{n}\left(c_{i}\right)}\right)+\frac{c}{q} \theta_{2}(c) \zeta_{n}\left(c_{i}\right) e^{\theta_{2}(c) \zeta_{n}\left(c_{i}\right)} \frac{\theta_{2}^{\prime}(c)}{\theta_{2}(c)}\right| h(x, c) \\
& +\left|H^{\zeta_{n}}\left(\zeta_{n}\left(c_{i}\right), c_{i+1}\right)-\frac{c}{q}\left(1-e^{\theta_{2}(c) \zeta_{n}\left(c_{i}\right)}\right)\right|\left|\partial_{c} h(x, c)\right| \\
\leq & K_{1}^{2}+K_{1}^{3}\left|\partial_{c} h(x, c)\right| .
\end{aligned}
$$

Calling $y=\zeta_{n}\left(c_{i}\right)>0$ and $\rho=\frac{x}{\zeta_{n}\left(c_{i}\right)} \in(0,1)$, we obtain

$$
\begin{aligned}
\partial_{c} h(x, c)= & T(\rho, y, c) \\
= & \frac{\left(\theta_{1}^{\prime}(c) \rho y e^{\theta_{1}(c) \rho y}-\theta_{2}^{\prime}(c) \rho y e^{\theta_{2}(c) \rho y}\right)\left(e^{\theta_{1}(c) y}-e^{\theta_{2}(c) y}\right)}{\left(e^{\theta_{1}(c) y}-e^{\theta_{2}(c) y}\right)^{2}} \\
& -\frac{\left(\theta_{1}^{\prime}(c) y e^{\theta_{1}(c) y}-\theta_{2}^{\prime}(c) y e^{\theta_{2}(c) y}\right)\left(e^{\theta_{1}(c) \rho y}-e^{\theta_{2}(c) \rho y}\right)}{\left(e^{\theta_{1}(c) y}-e^{\theta_{2}(c) y}\right)^{2}}
\end{aligned}
$$

Now, on the one hand,

$$
T(0, y, c)=T(1, y, c)=0, \lim _{y \rightarrow 0} T(\rho, y, c)=0
$$

and on the other hand, taking $\varepsilon>0$, and $y \geq \varepsilon$, there exists $K_{1}^{4}>0$ such that

$$
\begin{aligned}
& T(\rho, y, c)=-y(1-\rho) \theta_{1}(c) e^{-y(1-\rho) \theta_{1}(c)} \frac{\theta_{1}^{\prime}(c)}{\left(1-e^{\left(\theta_{2}(c)-\theta_{1}(c)\right) y}\right)^{2} \theta_{1}(c)} \\
&+\theta_{1}(c) y e^{-\theta_{1}(c) y} \frac{e^{\theta_{2}(c) \rho y}-\rho e^{\left((\rho-1) \theta_{1}(c)+\theta_{2}(c)\right) y} \frac{\theta}{1}_{\prime}(c)}{\theta_{1}(c)} \\
&\left(1-e^{\left.\left(\theta_{2}(c)-\theta_{1}(c)\right) y\right)^{2}}\right. \\
&+\theta_{1}(c) y e^{-\theta_{1}(c) y} \frac{e^{\left((\rho-1) \theta_{1}(c)+\theta_{2}(c)\right) y}+e^{\left((\rho+1) \theta_{2}(c)-\theta_{1}(c)\right) y}(\rho-1)-\rho e^{\theta_{2}(c) \rho y}}{\left(1-e^{\left.\left(\theta_{2}(c)-\theta_{1}(c)\right) y\right)^{2}}\right.} \frac{\theta_{2}^{\prime}(c)}{\theta_{1}(c)} \\
& \leq K_{1}^{4},
\end{aligned}
$$

because $s e^{-s}$ is bounded for $s \geq 0$. So we get 40 and finally the result.

## 6 Numerical examples

Let us finally consider a numerical illustration for the case $\mu=4, \sigma=2$ and $q=0.1$ for $S=[0, \bar{c}]$. In order to obtain the corresponding optimal value function $V^{S}$, we proceed as follows:

1. We obtain $\bar{\zeta}$ solving numerically the ordinary differential equation 31 with boundary condition (30), using the Euler method.
2. We check that the $\bar{\zeta}$-value function $W^{\bar{\zeta}}$ defined in 27) satisfies the conditions of Proposition 5.15. Hence $W^{\bar{\zeta}}$ is the optimal value function $V^{S}$.

Let us first consider the case $\bar{c}=4$ (i.e. the maximal allowed dividend rate is the drift of the surplus process $X_{t}$ ). Figure 1 a depicts $V^{S}(x, 0)$ as a function of initial capital $x$ together with the value function $V_{N R}(x)$ of the classical dividend problem without ratcheting constraint, for which the optimal strategy is a threshold strategy of not paying any dividends when the surplus level is below $b^{*}$ and pay dividends at rate $\bar{c}$ above $b^{*}$. Recall from Asmussen and Taksar [7] or also Gerber and Shiu [20] that in the notation of the present paper

$$
V_{N R}(x)= \begin{cases}\frac{\bar{c}}{q} \frac{e^{\theta_{1}(0) x}-e^{\theta_{2}(0) x}}{\theta_{1}\left(0 e^{\theta_{1}(0) b^{*}}-\theta_{2}(0) e^{\theta_{2}(0) b^{*}},\right.} & 0 \leq x \leq b^{*}, \\ \frac{\bar{c}}{q}+e^{\theta_{2}(\bar{c})\left(x-b^{*}\right)} / \theta_{2}(\bar{c}), & x \geq b^{*}\end{cases}
$$

with optimal threshold

$$
b^{*}=\frac{1}{\theta_{1}(0)-\theta_{2}(0)} \log \frac{\theta_{2}(0)\left(\theta_{2}(0)-\theta_{2}(\bar{c})\right)}{\theta_{1}(0)\left(\theta_{1}(0)-\theta_{2}(\bar{c})\right)}
$$

One observes that for both small and large initial capital $x$ the efficiency loss when introducing the ratcheting constraint is very small, only for intermediate values of $x$ the resulting expected discounted dividends are significantly smaller, but even there the relative efficiency loss is not big (see Figure 2a for a plot of this difference). We also compare $V^{S}(x, 0)$ in Figure 1a with the optimal value function

$$
V_{1}(x):=V^{0}(x) \text { for } S=\{0, \bar{c}\}
$$

of the further constrained one-step ratcheting strategy, where only once during the lifetime of the process the dividend rate can be increased from 0 to $\bar{c}$. That latter case was studied in detail in [3], where it was also shown that the optimal threshold level $b_{R}^{*}$ for that switch is exactly the one for which the resulting expected discounted dividends match with the ones of a threshold strategy underlying $V_{N R}$, but at the (for the latter problem non-optimal) threshold $b_{R}^{*}$. We observe that the performance of this simple one-step ratcheting is already remarkably close to the one of the overall optimal ratcheting strategy represented by $V^{S}(x, 0)$ (see also Figure 2 b for a plot of the difference). A similar effect had already been observed for the optimal ratcheting in the Cramér-Lundberg model (cf. [1]).

(a) $V^{S}(x, 0)$ (black) together with $V_{N R}(x)$ (blue) and $V_{1}(x)$ (red)

(b) Optimal curve $(\zeta(c), c)$ (black) together with $b^{*}$ (blue) and $b_{R}^{*}$ (red)

Figure 1: $\bar{c}=4$


Figure 2: $\bar{c}=4$

Figure 1 b depicts the optimal ratcheting curve $(\zeta(c), c)$ underlying $V^{S}(x, 0)$ for this example together with the optimal threshold $b^{*}$ of the unconstrained dividend problem and the optimal switching barrier $b_{R}^{*}$ for the one-step ratcheting strategy. One sees that the irreversibility of the dividend rate increase in the ratcheting case leads to a rather conservative behavior of not starting any (even not small) dividend payments until the surplus level is above the optimal threshold level $b^{*}$ underlying the non-constrained dividend problem. On the other hand, the one-step ratcheting strategy with optimal switching barrier $b_{R}^{*}$ roughly in the middle of the optimal curve already leads to a remarkably good approximation (lower bound) for the performance of the overall optimal ratcheting strategy.

In Figures 3 and 4 we give the analogous plots for the case $\bar{c}=8$, so that the maximal dividend rate is twice as large as the drift $\mu$ of the uncontrolled risk process. The global picture is quite similar, also in this case the efficiency loss introduced by ratcheting is more pronounced and present also for larger initial capital $x$. Also, the further efficiency loss by restricting to a simple one-step ratcheting strategy is considerably larger for not too large $x$. Finally, in that case the first increase of dividends already happens for surplus values (slightly) smaller than the optimal threshold $b^{*}$ of the unconstrained case.

## 7 Conclusion

In this paper we studied and solved the problem of finding optimal dividend strategies in a Brownian risk model, when the dividend rate can not be decreased over time. We showed that


Figure 3: $\bar{c}=8$


Figure 4: $\bar{c}=8$
the value function is the unique viscosity solution of a two-dimensional Hamilton-Jacobi-Bellman equation and it can be approximated arbitrarily closely by threshold strategies for finitely many possible dividend rates, which are established to be optimal in their discrete setting. We used calculus of variation techniques to identify the optimal curve that separates the state space into a change and a non-change region and provided partial results for the overall optimality of this strategy (which can be seen as a two-dimensional analogue of the optimality of dividend threshold strategies in the one-dimensional diffusion setting without the ratcheting constraint). In contrast to [2], the same analysis is applicable for all finite levels of maximal dividend rate $\bar{c}$, i.e. also if the latter exceeds the drift $\mu$. We also gave some numerical examples determining the optimal curve strategy. These results illustrate that the ratcheting constraint does not reduce the efficiency of the optimal dividend strategy substantially and that, much as in the compound Poisson setting, the simpler strategy of only stepping up the dividend rate once during the lifetime of the process is surprisingly close to optimal in absolute terms. In terms of a possible direction of future research, as mentioned in Section 5 we conjecture that a curve strategy dividing the state space into a change and a non-change region is optimal in full generality for the diffusion model, and it remains open to formally prove the latter. Furthermore, it could be interesting to extend the results of the present paper to the case where the dividend rate may be decreased by a certain percentage of its current value (see e.g. [5) or to place the dividend consumption pattern into a general habit formation framework (see e.g. [6] for an interesting related paper in a deterministic setup).

## 8 Appendix

Proof of Proposition 2.3. By Proposition 2.2. we have

$$
\begin{equation*}
0 \leq V^{S}\left(x_{2}, c_{1}\right)-V^{S}\left(x_{1}, c_{2}\right) \tag{41}
\end{equation*}
$$

for all $0 \leq x_{1} \leq x_{2}$ and $c_{1}, c_{2} \in S$ with $c_{1} \leq c_{2}$.
Let us show now, that there exists $K_{1}>0$ such that

$$
\begin{equation*}
V^{S}\left(x_{2}, c\right)-V^{S}\left(x_{1}, c\right) \leq K_{1}\left(x_{2}-x_{1}\right) \tag{42}
\end{equation*}
$$

for all $0 \leq x_{1} \leq x_{2}$. Take $\varepsilon>0$ and $C \in \Pi_{x_{2}, c}^{S}$ such that

$$
\begin{equation*}
J\left(x_{2} ; C\right) \geq V^{S}\left(x_{2}, c\right)-\varepsilon \tag{43}
\end{equation*}
$$

the associated control process is given by

$$
X_{t}^{C}=x_{2}+\int_{0}^{t}\left(\mu-C_{s}\right) d s+W_{t}
$$

Let $\tau$ be the ruin time of the process $X_{t}^{C}$. Define $\widetilde{C} \in \Pi_{x_{1}, c}^{S}$ as $\widetilde{C}_{t}=C_{t}$ and the associated control process

$$
X_{t}^{\widetilde{C}}=x_{1}+\int_{0}^{t}\left(\mu-C_{s}\right) d s+W_{t}
$$

Let $\widetilde{\tau} \leq \tau$ be the ruin time of the process $X_{t}^{\widetilde{C}}$; it holds that $X_{t}^{C}-X_{t}^{\widetilde{C}}=x_{2}-x_{1}$ for $t \leq \widetilde{\tau}$. Hence we have

$$
\begin{align*}
V^{S}\left(x_{2}, c\right)-V^{S}\left(x_{1}, c\right) & \leq J\left(x_{2} ; C\right)-J\left(x_{1} ; \widetilde{C}\right)+\varepsilon \\
& \leq V^{S}\left(x_{2}-x_{1}, 0\right)+\varepsilon \\
& \leq V_{N R}\left(x_{2}-x_{1}\right)+\varepsilon  \tag{44}\\
& \leq K_{1}\left(x_{2}-x_{1}\right)+\varepsilon
\end{align*}
$$

So, by Remark 2.1. we have 42 with $K_{1}=V_{N R}^{\prime}(0)$.
Let us show now that, given $c_{1}, c_{2} \in S$ with $c_{1} \leq c_{2}$, there exists $K_{2}>0$ such that

$$
\begin{equation*}
V^{S}\left(x, c_{1}\right)-V^{S}\left(x, c_{2}\right) \leq K_{2}\left(c_{2}-c_{1}\right) \tag{45}
\end{equation*}
$$

Take $\varepsilon>0$ and $C \in \Pi_{x, c_{1}}^{S}$ such that

$$
\begin{equation*}
J(x ; C) \geq V^{S}\left(x, c_{1}\right)-\varepsilon, \tag{46}
\end{equation*}
$$

define the stopping time

$$
\begin{equation*}
\widehat{T}=\min \left\{t: C_{t} \geq c_{2}\right\} \tag{47}
\end{equation*}
$$

and denote $\tau$ the ruin time of the process $X_{t}^{C}$. Let us consider $\widetilde{C} \in \Pi_{x, c_{2}}^{S}$ as $\widetilde{C}_{t}=c_{2} I_{t<\widehat{T}}+$ $C_{t} I_{t \geq \widehat{T}}$; denote by $X_{t}^{\widetilde{C}}$ the associated controlled surplus process and by $\bar{\tau} \leq \tau$ the corresponding ruin time. We have that $\widetilde{C}_{s}-C_{s} \leq c_{2}-c_{1}$ and so $X_{\bar{\tau}}^{C}=X_{\bar{\tau}}^{C}-X_{\widetilde{C}}^{\widetilde{C}} \leq\left(c_{2}-c_{1}\right) \bar{\tau}$, which implies

$$
\int_{\bar{\tau}}^{\tau} C_{s} e^{-q(s-\bar{\tau})} d s \leq V_{N R}\left(\left(c_{2}-c_{1}\right) \bar{\tau}\right) .
$$

Hence, we can write,

$$
\begin{align*}
V^{S}\left(x, c_{1}\right)-V^{S}\left(x, c_{2}\right) & \leq J(x ; C)+\varepsilon-J(x ; \widetilde{C}) \\
& =\mathbb{E}\left[\int_{0}^{\bar{\tau}}\left(C_{s}-\widetilde{C}_{s}\right) e^{-q s} d s\right]+\mathbb{E}\left[\int_{\bar{\tau}}^{\tau} C_{s} e^{-q s} d s\right]+\varepsilon \\
& \leq 0+\mathbb{E}\left[\int_{\bar{\tau}}^{\tau} C_{s} e^{-q s} d s\right]+\varepsilon  \tag{48}\\
& \leq E\left[e^{-q \bar{\tau}} \int_{\bar{\tau}}^{\tau} C_{s} e^{-q(s-\bar{\tau})} d s\right]+\varepsilon \\
& \leq K_{1} E\left[e^{-q \bar{\tau}} \bar{\tau}\left(c_{2}-c_{1}\right)\right]+\varepsilon \\
& \leq K_{2}\left(c_{2}-c_{1}\right)+\varepsilon .
\end{align*}
$$

So, we deduce 45 , taking $K_{2}=K_{1} \max _{t>0}\left\{e^{-q t} t\right\}$. We conclude the result from (41), 42) and (45).

Proof of Proposition 3.1. Let us show first that $V$ is a viscosity supersolution in $(0, \infty) \times[\underline{c}, \bar{c})$ By Proposition 2.2, $\partial_{c} \bar{V} \leq 0$ in $(0, \infty) \times[\underline{c}, \bar{c})$ in the viscosity sense.

Consider now $(x, c) \in(0, \infty) \times[\underline{c}, \bar{c})$ and the admissible strategy $C \in \Pi_{x, c}^{S}$, which pays dividends at constant rate $c$ up to the ruin time $\tau$. Let $X_{t}^{C}$ be the corresponding controlled surplus process and suppose that there exists a test function $\varphi$ for supersolution (8) at $(x, c)$. Using Lemma 2.1, we get for $h>0$

$$
\begin{aligned}
\varphi(x, c) & =V(x, c) \\
& \left.\geq \mathbb{E}\left[\int_{0}^{\tau \wedge h} e^{-q s} c d s\right]+\mathbb{E}\left[e^{-q(\tau \wedge h)} \varphi\left(X_{\tau \wedge h}^{C}, c\right)\right)\right] .
\end{aligned}
$$

Hence, using Itô's formula

$$
\begin{aligned}
0 & \geq \mathbb{E}\left[\int_{0}^{\tau \wedge h} e^{-q s} c d s\right]+\mathbb{E}\left[I_{\tau>h}\left(e^{-q h} \varphi\left(X_{s}^{C}, c\right)-\varphi(x, c)\right)\right]-\varphi(x, c) \mathbb{P}(h>\tau) \\
& =\mathbb{E}\left[\int_{0}^{\tau \wedge h} e^{-q s} c d s\right]+\mathbb{E}\left[I_{\tau>h} \int_{0}^{h} e^{-q s}\left(\frac{\sigma^{2}}{2} \partial_{x x} \varphi\left(X_{s}^{C}, c\right)+\partial_{x} \varphi\left(X_{s}^{C}, c\right)(\mu-c)-q \varphi\left(X_{s}^{C}, c\right)\right) d s\right]-\varphi(x, c) \mathbb{P}(h>
\end{aligned}
$$

So, dividing by $h$ and taking $h \rightarrow 0^{+}$, we get $\mathcal{L}^{c}(\varphi)(x, c) \leq 0$;, so that $V$ is a viscosity supersolution at $(x, c)$.

Let us prove now that $V$ it is a viscosity subsolution in $(0, \infty) \times[\underline{c}, \bar{c})$. Assume first that $V$ is not a subsolution of (8) at $(x, c) \in(0, \infty) \times[\underline{c}, \bar{c})$. Then there exist $\varepsilon>0,0<h<$ $\min \{x / 2, \bar{c}-c\}$ and a (2,1)-differentiable function $\psi$ with $\psi(x, c)=V(x, c)$ such that $\psi \geq V$,

$$
\begin{equation*}
\max \left\{\mathcal{L}^{c}(\psi)(y, d), \partial_{c} \psi(y, d)\right\} \leq-q \varepsilon<0 \tag{49}
\end{equation*}
$$

for $(y, d) \in[x-h, x+h] \times[c, c+h]$ and

$$
\begin{equation*}
V(y, d) \leq \psi(y, d)-\varepsilon \tag{50}
\end{equation*}
$$

for $(y, d) \notin[x-h, x+h] \times[c, c+h]$. Consider the controlled risk process $X_{t}$ corresponding to an admissible strategy $C \in \Pi_{x, c}^{S}$ and define

$$
\tau^{*}=\inf \left\{t>0: \quad\left(X_{t}, C_{t}\right) \notin[x-h, x+h] \times[c, c+h]\right\} .
$$

Since $C_{t}$ is non-decreasing and right-continuous, it can be written as

$$
\begin{equation*}
C_{t}=c+\int_{0}^{t} d C_{s}^{c o}+\sum_{\substack{C_{s} \neq C_{s^{-}} \\ 0 \leq s \leq t}}\left(C_{s}-C_{s^{-}}\right), \tag{51}
\end{equation*}
$$

where $C_{s}^{c o}$ is a continuous and non-decreasing function.
Take a (2,1)-differentiable function $\psi:(0, \infty) \times[\underline{c}, \bar{c}] \rightarrow[0, \infty)$. Using the expression (51) and the change of variables formula (see for instance [24]), we can write

$$
\begin{align*}
& e^{-q \tau^{*}} \psi\left(X_{\tau^{*}}^{C}, C_{\tau^{*}}\right)-\psi(x, c) \\
&= \int_{0}^{\tau^{*}} e^{-q s} \partial_{x} \psi\left(X_{s}^{C}, C_{s^{-}}\right)\left(\mu-C_{s^{-}}\right) d s+\int_{0}^{\tau^{*}} e^{-q s} \partial_{c} \psi\left(X_{s}^{C}, C_{s^{-}}\right) d C_{s}^{c o} \\
& \quad+\sum_{C_{s} \neq C_{s}} e^{-q s}\left(C_{s}-C_{s^{-}}\right) \partial_{c} \psi\left(X_{s}^{C}, C_{s^{-}}\right) \\
& \quad+\int_{0}^{\tau^{\tau^{*}} \leq s \leq \tau^{*}} e^{-q s}\left(-q \psi\left(X_{s}^{C}, C_{s^{-}}\right)+\frac{\sigma^{2}}{2} \partial_{x x} \psi\left(X_{s}^{C}, C_{s^{-}}\right)\right) d s+\int_{0}^{\tau^{*}} e^{-q s} \partial_{x} \psi\left(X_{s}^{C}, C_{s^{-}}\right) \sigma d W_{s} . \tag{52}
\end{align*}
$$

Hence, from (49), we can write

$$
\begin{aligned}
& \mathbb{E}\left[e^{-q \tau^{*}} \psi\left(X_{\tau^{*}}^{C}, C_{\tau^{*}}\right)\right]-\psi(x, c) \\
&= \mathbb{E}\left[\int_{0}^{\tau^{*}} e^{-q s} \mathcal{L}^{C_{s}-}(\psi)\left(X_{s^{-}}^{C}, C_{s^{-}}\right) d s-\int_{0}^{\tau^{*}} e^{-q s} C_{s^{-}} d s\right] \\
&+\mathbb{E}\left[\int_{0}^{\tau^{*}} e^{-q s} \partial_{c} \psi\left(X_{s^{-}}^{C}, C_{s^{-}}\right) d C_{s}^{c}+\sum_{\substack{C_{s} \neq C_{s}-\\
0 \leq s \leq \tau^{*}}} e^{-q s}\left(C_{s}-C_{s^{-}}\right) \partial_{c} \psi\left(X_{s^{-}}^{C}, C_{s^{-}}\right)\right] \\
& \leq \mathbb{E}\left[\varepsilon\left(e^{-q \tau^{*}}-1\right)-\int_{0}^{\tau^{*}} e^{-q s} C_{s^{-}} d s-q \varepsilon\left(\int_{0}^{\tau^{*}} e^{-q s} d C_{s}\right)\right] .
\end{aligned}
$$

So, from 50)

$$
\begin{aligned}
& \mathbb{E}\left[e^{-q \tau^{*}} V\left(X_{\tau^{*}}^{C}, C_{\tau^{*}}\right)\right] \\
& \leq \mathbb{E}\left[\psi(x, c)-e^{-q \tau^{*}} \varepsilon\right]+\mathbb{E}\left[\psi\left(X_{\tau^{*}}^{C}, C_{\tau^{*}}\right) e^{-q \tau^{*}}-\psi(x, c)\right] \\
& \leq \psi(x, c)-\varepsilon-\mathbb{E}\left(\int_{0}^{\tau^{*}} e^{-q s} C_{s^{-}} d s\right) .
\end{aligned}
$$

Hence, using Lemma 2.1, we have that

$$
V(x, c)=\sup _{C \in \Pi_{x, c}^{S}} \mathbb{E}\left(\int_{0}^{\tau^{*}} e^{-q s} C_{s^{-}} d s+e^{-c \tau^{*}} V\left(X_{\tau^{*}}^{C}, C_{\tau^{*}}\right)\right) \leq \psi(x, c)-\varepsilon
$$

but this is a contradiction because we have assumed that $V(x, c)=\psi(x, c)$. So we have the result.

Proof of Lemma 3.2. A locally Lipschitz function $\bar{u}:[0, \infty) \times[\underline{c}, \bar{c}] \rightarrow \mathbb{R}$ is a viscosity supersolution of (8) at $(x, c) \in(0, \infty) \times(\underline{c}, \bar{c})$, if any test function $\varphi$ for supersolution at $(x, c)$ satisfies

$$
\begin{equation*}
\max \left\{\mathcal{L}^{c}(\varphi)(x, c), \partial_{c} \varphi(x, c)\right\} \leq 0, \tag{53}
\end{equation*}
$$

and a locally Lipschitz function $\underline{u}:[0, \infty) \times[\underline{c}, \bar{c}] \rightarrow \mathbb{R}$ is a viscosity subsolution of (8) at $(x, c) \in(0, \infty) \times(\underline{c}, \bar{c})$ if any test function $\psi$ for subsolution at $(x, c)$ satisfies

$$
\begin{equation*}
\max \left\{\mathcal{L}^{c}(\psi)(x, c), \partial_{c} \psi(x, c)\right\} \geq 0 \tag{54}
\end{equation*}
$$

Suppose that there is a point $\left(x_{0}, c_{0}\right) \in[0, \infty) \times(\underline{c}, \bar{c})$ such that $\underline{u}\left(x_{0}, c_{0}\right)-\bar{u}\left(x_{0}, c_{0}\right)>0$. Let us define $h(c)=1+e^{-c / \bar{c}}$ and

$$
\bar{u}^{s}(x, c)=\operatorname{sh}(c) \bar{u}(x, c)
$$

for any $s>1$. We have that $\varphi$ is a test function for supersolution of $\bar{u}$ at $(x, c)$ if and only if $\varphi^{s}=s h(c) \varphi$ is a test function for supersolution of $\bar{u}^{s}$ at $(x, c)$. We have

$$
\begin{equation*}
\mathcal{L}^{c}\left(\varphi^{s}\right)(x, c)=\operatorname{sh}(c) \mathcal{L}^{c}(\varphi)(x, c)+c(1-\operatorname{sh}(c))<0, \tag{55}
\end{equation*}
$$

and

$$
\begin{equation*}
\partial_{c} \varphi^{s}(x, c) \leq-\frac{s}{\bar{c}} \varphi(x, c) e^{-\frac{c}{c}}<0 \tag{56}
\end{equation*}
$$

for $\varphi(x, c)>0$. Take $s_{0}>1$ such that $\underline{u}\left(x_{0}, c_{0}\right)-\bar{u}^{s_{0}}\left(x_{0}, c_{0}\right)>0$. We define

$$
\begin{equation*}
M=\sup _{x \geq 0, c \leq c \leq \bar{c}}\left(\underline{u}(x, c)-\bar{u}^{s_{0}}(x, c)\right) . \tag{57}
\end{equation*}
$$

Since $\lim _{x \rightarrow \infty} \underline{u}(x, c) \leq \bar{c} / q \leq \lim _{x \rightarrow \infty} \bar{u}(x, c)$, there exist $b>x_{0}$ such that

$$
\begin{equation*}
\sup _{\underline{c} \leq c \leq \bar{c}} \underline{u}(x, c)-\bar{u}^{s_{0}}(x, c)<0 \text { for } x \geq b . \tag{58}
\end{equation*}
$$

From (58), we obtain that

$$
\begin{equation*}
0<\underline{u}\left(x_{0}, c_{0}\right)-\bar{u}^{s_{0}}\left(x_{0}, c_{0}\right) \leq M:=\max _{x \in[0, b], \underline{c} \leq c \leq \bar{c}}\left(\underline{u}(x, c)-\bar{u}^{s_{0}}(x, c)\right) . \tag{59}
\end{equation*}
$$

Call $\left(x^{*}, c^{*}\right):=\arg \max _{x \in[0, b], \underline{c} \leq c \leq \bar{c}}\left(\underline{u}(x, c)-\bar{u}^{s_{0}}(x, c)\right)$. Let us consider the set

$$
\mathcal{A}=\{(x, y, c, d): 0 \leq x \leq y \leq b, \underline{c} \leq c \leq \bar{c}, \underline{c} \leq d \leq \bar{c}\}
$$

and, for all $\lambda>0$, the functions

$$
\begin{align*}
& \Phi^{\lambda}(x, y, c, d)=\frac{\lambda}{2}(x-y)^{2}+\frac{\lambda}{2}(c-d)^{2}+\frac{2 m}{\lambda^{2}(y-x)+\lambda},  \tag{60}\\
& \Sigma^{\lambda}(x, y, c, d)=\underline{u}(x, c)-\bar{u}^{s_{0}}(y, d)-\Phi^{\lambda}(x, y, c, d) .
\end{align*}
$$

Calling $M^{\lambda}=\max _{A} \Sigma^{\lambda}$ and $\left(x_{\lambda}, y_{\lambda}, c_{\lambda}, d_{\lambda}\right)=\arg \max _{A} \Sigma^{\lambda}$, we obtain that $M^{\lambda} \geq \Sigma^{\lambda}\left(x^{*}, x^{*}, c^{*}, c^{*}\right)=$ $M-\frac{2 m}{\lambda}$, and so

$$
\begin{equation*}
\liminf _{\lambda \rightarrow \infty} M^{\lambda} \geq M \tag{61}
\end{equation*}
$$

There exists $\lambda_{0}$ large enough such that if $\lambda \geq \lambda_{0}$, then $\left(x_{\lambda}, y_{\lambda}, c_{\lambda}, d_{\lambda}\right) \notin \partial A$, the proof is similar to the one of Lemma 4.5 of [2].

Using the inequality

$$
\Sigma^{\lambda}\left(x_{\lambda}, x_{\lambda}, c_{\lambda}, c_{\lambda}\right)+\Sigma^{\lambda}\left(y_{\lambda}, y_{\lambda}, d_{\lambda}, d_{\lambda}\right) \leq 2 \Sigma^{\lambda}\left(x_{\lambda}, y_{\lambda}, c_{\lambda}, d_{\lambda}\right)
$$

we obtain that

$$
\lambda\left\|\left(x_{\lambda}-y_{\lambda}, c_{\lambda}-d_{\lambda}\right)\right\|_{2}^{2} \leq \underline{u}\left(x_{\lambda}, c_{\lambda}\right)-\underline{u}\left(y_{\lambda}, d_{\lambda}\right)+\bar{u}^{s_{0}}\left(x_{\lambda}, c_{\lambda}\right)-\bar{u}^{s_{0}}\left(y_{\lambda}, d_{\lambda}\right)+4 m\left(y_{\lambda}-x_{\lambda}\right) .
$$

Consequently

$$
\begin{equation*}
\lambda\left\|\left(x_{\lambda}-y_{\lambda}, c_{\lambda}-d_{\lambda}\right)\right\|_{2}^{2} \leq 6 m\left\|\left(x_{\lambda}-y_{\lambda}, c_{\lambda}-d_{\lambda}\right)\right\|_{2} . \tag{62}
\end{equation*}
$$

We can find a sequence $\lambda_{n} \rightarrow \infty$ such that $\left(x_{\lambda_{n}}, y_{\lambda_{n}}, c_{\lambda_{n}}, d_{\lambda_{n}}\right) \rightarrow(\hat{x}, \widehat{y}, \widehat{c}, \hat{d}) \in A$. From 62, we get that

$$
\begin{equation*}
\left\|\left(x_{\lambda_{n}}-y_{\lambda_{n}}, c_{\lambda_{n}}-d_{\lambda_{n}}\right)\right\|_{2} \leq 6 m / \lambda_{n}, \tag{63}
\end{equation*}
$$

which gives $\widehat{x}=\widehat{y}$ and $\widehat{c}=\widehat{d}$.
Since $\Sigma^{\lambda}(x, y, c, d)=\underline{u}(x, c)-\bar{u}^{s_{0}}(y, d)-\Phi^{\lambda}(x, y, c, d)$ reaches the maximum in $\left(x_{\lambda}, y_{\lambda}, c_{\lambda}, d_{\lambda}\right)$ in the interior of the set $A$, the function

$$
\psi(x, c)=\Phi^{\lambda}\left(x, y_{\lambda}, c, d_{\lambda}\right)-\Phi^{\lambda}\left(x_{\lambda}, y_{\lambda}, c_{\lambda}, d_{\lambda}\right)+\underline{u}\left(x_{\lambda}, c_{\lambda}\right)
$$

is a test for subsolution for $\underline{u}$ of the HJB equation at the point $\left(x_{\lambda}, c_{\lambda}\right)$. In addition, the function

$$
\varphi^{s_{0}}(y, d)=-\Phi^{\lambda}\left(x_{\lambda}, y, c_{\lambda}, d\right)+\Phi^{\lambda}\left(x_{\lambda}, y_{\lambda}, c_{\lambda}, d_{\lambda}\right)+\bar{u}^{s_{0}}\left(y_{\lambda}, d_{\lambda}\right)
$$

is a test for supersolution for $\bar{u}^{s_{0}}$ at $\left(y_{\lambda}, d_{\lambda}\right)$ and so

$$
\partial_{c} \varphi^{s_{0}}\left(y_{\lambda}, d_{\lambda}\right) \leq-\frac{s_{0}}{c_{2}} \varphi\left(y_{\lambda}, d_{\lambda}\right) e^{-\frac{c}{c_{2}}}<0
$$

(because $y_{\lambda}>0$ ). Hence, $\partial_{c} \psi\left(x_{\lambda}, c_{\lambda}\right)=\partial_{c} \varphi^{s_{0}}\left(y_{\lambda}, d_{\lambda}\right)<0$, and we have $\mathcal{L}^{c_{\lambda}}(\psi)\left(x_{\lambda}, c_{\lambda}\right) \geq 0$.
Assume first that the functions $\underline{u}(x, c)$ and $\bar{u}^{s_{0}}(y, d)$ are (2,1)-differentiable at $\left(x_{\lambda}, c_{\lambda}\right)$ and $\left(y_{\lambda}, d_{\lambda}\right)$ respectively. Since $\Sigma^{\lambda}$ defined in reaches a local maximum at $\left(x_{\lambda}, y_{\lambda}, c_{\lambda}, d_{\lambda}\right) \notin \partial A$, we have that

$$
\partial_{x} \Sigma^{\lambda}\left(x_{\lambda}, y_{\lambda}, c_{\lambda}, d_{\lambda}\right)=\partial_{y} \Sigma^{\lambda}\left(x_{\lambda}, y_{\lambda}, c_{\lambda}, d_{\lambda}\right)=0
$$

and so

$$
\begin{align*}
\partial_{x} \underline{u}\left(x_{\lambda}, c_{\lambda}\right) & =\partial_{x} \Phi^{\lambda}\left(x_{\lambda}, y_{\lambda}, c_{\lambda}, d_{\lambda}\right) \\
& =\lambda\left(x_{\lambda}-y_{\lambda}\right)+\frac{2 m}{\left(\lambda\left(y_{\lambda}-x_{\lambda}\right)+1\right)^{2}}  \tag{64}\\
& =-\partial_{y} \Phi^{\lambda}\left(x_{\lambda}, y_{\lambda}, c_{\lambda}, d_{\lambda}\right)=\partial_{x} \bar{u}^{s_{0}}\left(y_{\lambda}, d_{\lambda}\right)
\end{align*}
$$

Defining $A=\partial_{x x} \underline{u}\left(x_{\lambda}, c_{\lambda}\right)$ and $B=\partial_{x x} \bar{u}^{s_{0}}\left(y_{\lambda}, d_{\lambda}\right)$, we obtain

$$
\begin{aligned}
& \left(\begin{array}{ll}
\partial_{x x} \Sigma^{\lambda}\left(x_{\lambda}, y_{\lambda}, c_{\lambda}, d_{\lambda}\right) & \partial_{x y} \Sigma^{\lambda}\left(x_{\lambda}, y_{\lambda}, c_{\lambda}, d_{\lambda}\right) \\
\partial_{x y} \Sigma^{\lambda}\left(x_{\lambda}, y_{\lambda}, c_{\lambda}, d_{\lambda}\right) & \partial_{y y} \Sigma^{\lambda}\left(x_{\lambda}, y_{\lambda}, c_{\lambda}, d_{\lambda}\right)
\end{array}\right) \\
& =\left(\begin{array}{cc}
A-\partial_{x x} \Phi^{\lambda}\left(x_{\lambda}, y_{\lambda}, c_{\lambda}, d_{\lambda}\right) & -\partial_{x y} \Phi^{\lambda}\left(x_{\lambda}, y_{\lambda}, c_{\lambda}, d_{\lambda}\right) \\
-\partial_{x y} \Phi^{\lambda}\left(x_{\lambda}, y_{\lambda}, c_{\lambda}, d_{\lambda}\right) & -B-\partial_{y y} \Phi^{\lambda}\left(x_{\lambda}, y_{\lambda}, c_{\lambda}, d_{\lambda}\right)
\end{array}\right) \leq 0 .
\end{aligned}
$$

It is hence a negative semi-definite matrix, and

$$
\left(\begin{array}{cc}
A & 0 \\
0 & -B
\end{array}\right) \leq \partial_{x y} H\left(\Phi^{\lambda}\right)\left(x_{\lambda}, y_{\lambda}, c_{\lambda}, d_{\lambda}\right):=\left(\begin{array}{cc}
\partial_{x x} \Phi^{\lambda}\left(x_{\lambda}, y_{\lambda}, c_{\lambda}, d_{\lambda}\right) & \partial_{x y} \Phi^{\lambda}\left(x_{\lambda}, y_{\lambda}, c_{\lambda}, d_{\lambda}\right) \\
\partial_{x y} \Phi^{\lambda}\left(x_{\lambda}, y_{\lambda}, c_{\lambda}, d_{\lambda}\right) & \partial_{y y} \Phi^{\lambda}\left(x_{\lambda}, y_{\lambda}, c_{\lambda}, d_{\lambda}\right)
\end{array}\right) .
$$

In the case that $\underline{u}(x, c)$ and $\bar{u}^{s_{0}}(y, d)$ are not (2,1)-differentiable at $\left(x_{\lambda}, c_{\lambda}\right)$ and ( $y_{\lambda}, d_{\lambda}$ ) respectively, we can resort to a more general theorem to get a similar result. Using Theorem 3.2 of Crandall, Ishii and Lions [14, it can be proved that for any $\delta>0$, there exist real numbers $A_{\delta}$ and $B_{\delta}$ such that

$$
\left(\begin{array}{cc}
A_{\delta} & 0  \tag{65}\\
0 & -B_{\delta}
\end{array}\right) \leq \partial_{x y} H\left(\Phi^{\lambda}\right)\left(x_{\lambda}, y_{\lambda}, c_{\lambda}, d_{\lambda}\right)+\delta\left(\partial_{x y} H\left(\Phi^{\lambda}\right)\left(x_{\lambda}, y_{\lambda}, c_{\lambda}, d_{\lambda}\right)\right)^{2}
$$

and

$$
\begin{gather*}
\frac{\sigma^{2}}{2} A_{\delta}+\left(\mu-c_{\lambda}\right) \partial_{x} \psi\left(x_{\lambda}, c_{\lambda}\right)-q \psi\left(x_{\lambda}, c_{\lambda}\right)+c_{\lambda} \geq 0  \tag{66}\\
\frac{\sigma^{2}}{2} B_{\delta}+\left(\mu-d_{\lambda}\right) \partial_{x} \varphi^{s_{0}}\left(y_{\lambda}, d_{\lambda}\right)-q \varphi^{s_{0}}\left(y_{\lambda}, d_{\lambda}\right)+d_{\lambda} \leq 0
\end{gather*}
$$

The expression (65) implies that $A_{\delta}-B_{\delta} \leq 0$ because

$$
\partial_{x y} H\left(\Phi^{\lambda}\right)\left(x_{\lambda}, y_{\lambda}, c_{\lambda}, d_{\lambda}\right)=\partial_{x x} \Phi^{\lambda}\left(x_{\lambda}, y_{\lambda}, c_{\lambda}, d_{\lambda}\right)\left(\begin{array}{cc}
1 & -1 \\
-1 & 1
\end{array}\right)
$$

and

$$
\left(\partial_{x y} H\left(\Phi^{\lambda}\right)\left(x_{\lambda}, y_{\lambda}, c_{\lambda}, d_{\lambda}\right)\right)^{2}=2\left(\partial_{x x} \Phi^{\lambda}\left(x_{\lambda}, y_{\lambda}, c_{\lambda}, d_{\lambda}\right)\right)^{2}\left(\begin{array}{cc}
1 & -1 \\
-1 & 1
\end{array}\right)
$$

Therefore,

$$
\begin{aligned}
A_{\delta}-B_{\delta} & =\left(\begin{array}{ll}
1 & 1
\end{array}\right)\left(\begin{array}{cc}
A_{\delta} & 0 \\
0 & -B_{\delta}
\end{array}\right)\binom{1}{1} \\
& \leq\left(\begin{array}{ll}
1 & 1
\end{array}\right)\left(\partial_{x y} H\left(\Phi^{\lambda}\right)\left(x_{\lambda}, y_{\lambda}, c_{\lambda}, d_{\lambda}\right)+\delta\left(\partial_{x y} H\left(\Phi^{\lambda}\right)\left(x_{\lambda}, y_{\lambda}, c_{\lambda}, d_{\lambda}\right)\right)^{2}\right)\binom{1}{1} \\
& =0
\end{aligned}
$$

And so, since $\varphi^{s_{0}}\left(y_{\lambda}, d_{\lambda}\right)=\bar{u}^{s_{0}}\left(y_{\lambda}, d_{\lambda}\right), \psi\left(x_{\lambda}, c_{\lambda}\right)=\underline{u}\left(x_{\lambda}, c_{\lambda}\right)$ and

$$
\partial_{x} \varphi^{s_{0}}\left(y_{\lambda}, d_{\lambda}\right)=-\partial_{y} \Phi^{\lambda}\left(x_{\lambda}, y_{\lambda}, c_{\lambda}, d_{\lambda}\right)=\partial_{x} \Phi^{\lambda}\left(x_{\lambda}, y_{\lambda}, c_{\lambda}, d_{\lambda}\right)=\partial_{x} \psi\left(x_{\lambda}, c_{\lambda}\right)
$$

we obtain

$$
\begin{align*}
\underline{u}\left(x_{\lambda}, c_{\lambda}\right)-\bar{u}^{s_{0}}\left(y_{\lambda}, d_{\lambda}\right)= & \psi\left(x_{\lambda}, c_{\lambda}\right)-\varphi^{s_{0}}\left(y_{\lambda}, d_{\lambda}\right) \\
\leq & \frac{\sigma^{2}}{2 q}\left(A_{\delta}-B_{\delta}\right) \\
& +\left(\frac{c_{\lambda}}{q}-\frac{d_{\lambda}}{q}\right)\left(1-\partial_{x} \Phi^{\lambda}\left(x_{\lambda}, y_{\lambda}, c_{\lambda}, d_{\lambda}\right)\right)  \tag{67}\\
\leq & \left(\frac{c_{\lambda}}{q}-\frac{d_{\lambda}}{q}\right)\left(1-\lambda\left(x_{\lambda}-y_{\lambda}\right)-\frac{2 m}{\left(\lambda\left(y_{\lambda}-x_{\lambda}\right)+1\right)^{2}}\right) .
\end{align*}
$$

Hence, from (63) and (61), we get

$$
\begin{aligned}
0 & <M \leq \liminf _{\lambda \rightarrow \infty} M_{\lambda} \leq \lim _{n \rightarrow \infty} M_{\lambda_{n}}=\lim _{n \rightarrow \infty} \Sigma^{\lambda_{n}}\left(x_{\lambda_{n}}, y_{\lambda_{n}}, c_{\lambda_{n}}, d_{\lambda_{n}}\right)=\underline{u}(\widehat{x}, \widehat{c})-\bar{u}^{s_{0}}(\widehat{x}, \widehat{c}) \\
& \leq \lim _{n \longrightarrow \infty}\left(\frac{c_{\lambda_{n}}}{q}-\frac{d_{\lambda_{n}}}{q}\right)\left(1-\lambda_{n}\left(x_{\lambda_{n}}-y_{\lambda_{n}}\right)-\frac{2 m}{\left(\lambda_{n}\left(y_{\lambda_{n}}-x_{\lambda_{n}}\right)+1\right)^{2}}\right) \\
& \leq \lim _{n \longrightarrow \infty}\left|\frac{c_{\lambda_{n}}}{q}-\frac{d_{\lambda_{n}}}{q}\right|\left(1+\lambda_{n}\left\|\left(x_{\lambda_{n}}-y_{\lambda_{n}}, c_{\lambda_{n}}-d_{\lambda_{n}}\right)\right\|_{2}+\frac{2 m}{\left(\lambda_{n}\left(y_{\lambda_{n}}-x_{\lambda_{n}}\right)+1\right)^{2}}\right) \\
& \leq \lim _{n \longrightarrow \infty}\left|\frac{c_{\lambda_{n}}}{q}-\frac{d_{\lambda_{n}}}{q}\right|(1+8 m)=0 .
\end{aligned}
$$

This is a contradiction and so we get the result.

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